GCE AS/A LEVEL – NEW

2305U10-1

S19-2305U10-1

MONDAY, 13 MAY 2019 – AFTERNOON

FURTHER MATHEMATICS – AS unit 1
FURTHER PURE MATHEMATICS A

1 hour 30 minutes

ADDITIONAL MATERIALS

In addition to this examination paper, you will need:

- a WJEC pink 16-page answer booklet;
- a Formula Booklet;
- a calculator.

INSTRUCTIONS TO CANDIDATES

Use black ink or black ball-point pen.

Answer **all** questions.

Sufficient working must be shown to demonstrate the **mathematical** method employed.

Answers without working may not gain full credit.

Unless the degree of accuracy is stated in the question, answers should be rounded appropriately.

INFORMATION FOR CANDIDATES

The maximum mark for this paper is 70.

The number of marks is given in brackets at the end of each question or part-question.

You are reminded of the necessity for good English and orderly presentation in your answers.

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Reminder: *Sufficient working must be shown to demonstrate the **mathematical** method employed.*

1. The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 3 & 7 \\ -2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix}$.

The matrix **X** is such that **AX = B**. Showing all your working, find the matrix **X**. [6]

2. The position vectors of the points *A*, *B*, *C*, *D* are given by

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{c} = 7\mathbf{i} - 3\mathbf{k}, \qquad \mathbf{d} = -3\mathbf{i} - \mathbf{j} - 5\mathbf{k},$$

respectively.

(a) Find the vector equations of the lines *AB* and *CD*. [3]

(b) Determine whether or not the lines *AB* and *CD* are perpendicular. [4]

3. The complex numbers z and w are represented by the points *Z* and *W* in an Argand diagram.

The complex number z is such that $|z| = 6$ and $\arg z = \frac{\pi}{3}$.

The point *W* is a 90° clockwise rotation, about the origin, of the point *Z* in the Argand diagram.

(a) Express z and w in the form $x + \mathrm{i}y$. [3]

(b) Find the complex number $\frac{z}{w}$. [4]

4. Prove, by mathematical induction, that $9^n + 15$ is a multiple of 8 for all positive integers n. [7]

5. Given that $x = -\frac{1}{2}$ and $x = -3$ are two roots of the equation

$$2x^4 - x^3 - 15x^2 + 23x + 15 = 0,$$

find the remaining roots. [6]

6. The complex number z is represented by the point $P(x, y)$ in an Argand diagram.
Given that

$$|z - 1| = |z - 2\mathrm{i}|,$$

show that the locus of *P* is a straight line. [3]

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7. *(a)* Find an expression for $\sum_{r=1}^{2m}(r+2)^2$ in the form $\frac{1}{3}m(am^2+bm+c)$, where a, b, c are integers whose values are to be determined. [4]

(b) Hence, calculate $\sum_{r=11}^{20}(r+2)^2$. [4]

8. The plane Π contains the three points $A(3, 5, 6)$, $B(5, -1, 7)$ and $C(-1, 7, 0)$.

Find the vector equation of the plane Π in the form $\mathbf{r.n} = d$.
Express this equation in Cartesian form. [9]

9. The complex numbers z and w are represented by the points $P(x, y)$ and $Q(u, v)$ respectively in Argand diagrams and

$$w = z^2 - 1.$$

(a) Show that $v = 2xy$ and obtain an expression for u in terms of x and y. [4]

(b) The point P moves along the line $y = 3x$. Find the equation of the locus of Q. [4]

10. The quadratic equation $px^2 + qx + r = 0$ has roots α and β, where p, q, r are non-zero constants.

(a) A cubic equation is formed with roots α, β, $\alpha + \beta$.

Find the cubic equation with coefficients expressed in terms of p, q, r. [6]

(b) Another quadratic equation $px^2 - qx - r = 0$ has roots 2α and γ.

Show that $\beta = -2\gamma$. [3]

END OF PAPER

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Surname	StemUp Model Solution
Answer booklet	Write the question number in the left-hand margin at the start of each answer.

Question number

1

$AX = B \quad \Rightarrow \quad X = A^{-1}B$

$\det A = 3 \times 0 - 7 \times (-2) = 14$

$A^{-1} = \frac{1}{14}\begin{pmatrix} 0 & -7 \\ 2 & 3 \end{pmatrix}$

$X = \frac{1}{14}\begin{pmatrix} 0 & -7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 0 & -28 \\ 10 & 14 \end{pmatrix}$

$X = \begin{pmatrix} 0 & -2 \\ 5/7 & 1 \end{pmatrix}$

2 (a)

AB: $r = a + \lambda(b - a)$

$r = 2i + 3j - k + \lambda(4j + 5k - 2i - 3j + k)$

$r = 2i + 3j - k + \lambda(-2i + j + 6k)$

CD: $r = c + \mu(d - c)$

$r = 7i - 3k + \mu(-3i - j - 5k - 7i + 3k)$

$r = 7i - 3k + \mu(-10i - j - 2k)$

2 (b)

Direction vectors: $-2i + j + 6k$ and $-10i - j - 2k$

$(-2i + j + 6k).(-10i - j - 2k) = 20 - 1 - 12 = 7$

Because $7 \neq 0$, the scalar product is not zero,

so the lines AB and CD are not perpendicular.

3 (a)

$z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3 + 3\sqrt{3}\,i$

Clockwise rotation of 90°: $\arg w = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$, $|w| = 6$

Question number

$w = 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 3\sqrt{3} - 3i$

3 (b)

$\frac{z}{w} = \frac{3 + 3\sqrt{3}\,i}{3\sqrt{3} - 3i} = \frac{(3 + 3\sqrt{3}\,i)(3\sqrt{3} + 3i)}{(3\sqrt{3} - 3i)(3\sqrt{3} + 3i)}$

$= \frac{9\sqrt{3} + 9i + 27i + 9\sqrt{3}\,i^2}{27 + 9} = \frac{9\sqrt{3} + 36i - 9\sqrt{3}}{36} = \frac{36i}{36}$

$\frac{z}{w} = i$

4

Let $P(n)$ be the proposition that $9^n + 15$ is a multiple of 8.

$n = 1$: $9^1 + 15 = 24 = 8 \times 3$, which is a multiple of 8,

so the proposition is true for $n = 1$.

Assume the proposition is true for $n = k$, i.e. $9^k + 15 = 8N$

for some integer N, so $9^k = 8N - 15$.

Consider $n = k + 1$:

$9^{k+1} + 15 = 9(9^k) + 15$

$= 9(8N - 15) + 15$

$= 72N - 135 + 15$

$= 72N - 120 = 8(9N - 15)$

72N and 120 are both multiples of 8, so $9^{k+1} + 15$ is a multiple of 8.

So if the proposition is true for $n = k$, it is also true for $n = k + 1$.

Since it is true for $n = 1$, by mathematical induction it is true

for all positive integers n.

5

$x = -\frac{1}{2}$ and $x = -3$ are roots, so $(2x + 1)$ and $(x + 3)$ are factors

$(2x + 1)(x + 3) = 2x^2 + 7x + 3$ is a quadratic factor

$2x^4 - x^3 - 15x^2 + 23x + 15 = (2x^2 + 7x + 3)(x^2 + bx + c)$

x^3: $-1 = 2b + 7 \Rightarrow b = -4$; constant: $15 = 3c \Rightarrow c = 5$

Question number

(check x: $7c + 3b = 35 - 12 = 23$ ✓)

$(2x^2 + 7x + 3)(x^2 - 4x + 5) = 0$

$x^2 - 4x + 5 = 0 \quad \Rightarrow \quad x = \frac{4 \pm \sqrt{(16 - 20)}}{2} = \frac{4 \pm 2i}{2}$

Remaining roots: $x = 2 + i$ and $x = 2 - i$

6 $z = x + iy$: $|x + iy - 1| = |x + iy - 2i|$

$|(x - 1) + iy| = |x + i(y - 2)|$

$\sqrt{(x - 1)^2 + y^2} = \sqrt{x^2 + (y - 2)^2}$

$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 4y + 4$

$-2x + 1 = -4y + 4$

$4y = 2x + 3$, i.e. $y = \frac{1}{2}x + \frac{3}{4}$, which is a straight line.

7 (a) $\sum_{r=1}^{2m} (r + 2)^2 = \sum_{r=1}^{2m} (r^2 + 4r + 4) = \sum_{r=1}^{2m} r^2 + 4\sum_{r=1}^{2m} r + \sum_{r=1}^{2m} 4$

$= \frac{1}{6}(2m)(2m + 1)(4m + 1) + 4 \times \frac{1}{2}(2m)(2m + 1) + 4(2m)$

$= \frac{1}{3}m(8m^2 + 6m + 1) + 8m^2 + 4m + 8m$

$= \frac{1}{3}m(8m^2 + 6m + 1 + 24m + 12 + 24)$

$= \frac{1}{3}m(8m^2 + 30m + 37)$

$a = 8, \quad b = 30, \quad c = 37$

7 (b) $\sum_{r=11}^{20} (r + 2)^2 = \sum_{r=1}^{20} (r + 2)^2 - \sum_{r=1}^{10} (r + 2)^2$

(using (a) with $m = 10$ for $2m = 20$, and $m = 5$ for $2m = 10$)

Question number

$= \frac{1}{3} \times 10 \times (8 \times 100 + 30 \times 10 + 37) - \frac{1}{3} \times 5 \times (8 \times 25 + 30 \times 5 + 37)$

$= \frac{1}{3} \times 10 \times 1137 - \frac{1}{3} \times 5 \times 387$

$= 3790 - 645$

$= 3145$

8

$AB = b - a = 2i - 6j + k, \quad AC = c - a = -4i + 2j - 6k$

Let $n = pi + qj + rk$ be normal to Π, so $AB.n = 0$ and $AC.n = 0$:

$2p - 6q + r = 0 \quad (1)$

$-4p + 2q - 6r = 0 \quad (2)$

From (1): $r = 6q - 2p$; sub into (2): $-4p + 2q - 36q + 12p = 0$

$8p = 34q \Rightarrow p = \frac{17}{4}q, \quad r = 6q - \frac{17}{2}q = -\frac{5}{2}q$

Let $q = 4$: $p = 17, \quad q = 4, \quad r = -10, \quad n = 17i + 4j - 10k$

$d = a.n = (3 \times 17) + (5 \times 4) + (6 \times -10) = 51 + 20 - 60 = 11$

(check C: $-17 + 28 + 0 = 11$ ✓)

Vector equation: $r.(17i + 4j - 10k) = 11$

Cartesian form: $17x + 4y - 10z = 11$

9 (a)

$u + iv = (x + iy)^2 - 1$

$u + iv = x^2 + 2ixy + i^2y^2 - 1 = x^2 - y^2 - 1 + 2xyi$

Comparing imaginary parts: $v = 2xy$ as required

Comparing real parts: $u = x^2 - y^2 - 1$

9 (b)

$y = 3x: \quad v = 2x(3x) = 6x^2$

$u = x^2 - 9x^2 - 1 = -8x^2 - 1$

Eliminating x: $x^2 = \frac{v}{6} \Rightarrow u = -8\left(\frac{v}{6}\right) - 1$

Question number

Locus of Q: $u = -\frac{4}{3}v - 1$ (or $3u + 4v + 3 = 0$)

10 (a) $\alpha + \beta = -\frac{q}{p}, \quad \alpha\beta = \frac{r}{p}$

Sum of roots: $\alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta) = -\frac{2q}{p}$

Sum of pairs: $\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta) = \alpha\beta + (\alpha + \beta)^2 = \frac{r}{p} + \frac{q^2}{p^2}$

Product: $\alpha\beta(\alpha + \beta) = \frac{r}{p} \times \left(-\frac{q}{p}\right) = -\frac{qr}{p^2}$

Cubic: $x^3 + \frac{2q}{p}x^2 + \left(\frac{r}{p} + \frac{q^2}{p^2}\right)x + \frac{qr}{p^2} = 0$

i.e. $p^2x^3 + 2pqx^2 + (pr + q^2)x + qr = 0$

10 (b) From $px^2 - qx - r = 0$: product of roots $2\alpha\gamma = -\frac{r}{p}$

From part (a): $\alpha\beta = \frac{r}{p}$, so $2\alpha\gamma = -\alpha\beta$

$\alpha \neq 0$ (since $r \neq 0$, $\alpha\beta \neq 0$), so dividing by α:

$2\gamma = -\beta \quad \therefore \quad \beta = -2\gamma$ as required