Surname	StemUp Model Solution	Centre Number	Candidate Number
First name(s)			2

GCE A LEVEL

S24-1305U60-1

1305U60-1

WEDNESDAY, 12 JUNE 2024 – AFTERNOON

FURTHER MATHEMATICS – A2 unit 6
FURTHER MECHANICS B

1 hour 45 minutes

For Examiner's use only		
Question	**Maximum Mark**	**Mark Awarded**
1	**11**	
2	**16**	
3	**13**	
4	**13**	
5	**13**	
6	**14**	
Total	**80**	

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ADDITIONAL MATERIALS

In addition to this examination paper, you will need:

- a Formula Booklet;
- a calculator.

INSTRUCTIONS TO CANDIDATES

Use black ink or black ball-point pen. Do not use gel pen or correction fluid.

You may use a pencil for graphs and diagrams only.

Write your name, centre number and candidate number in the spaces at the top of this page.

Answer **all** questions.

Take g as $9{\cdot}8\,\text{ms}^{-2}$.

Write your answers in the spaces provided in this booklet. If you run out of space, use the additional page(s) at the back of the booklet, taking care to number the question(s) correctly.

INFORMATION FOR CANDIDATES

The maximum mark for this paper is 80.

The number of marks is given in brackets at the end of each question or part-question.

Sufficient working must be shown to demonstrate the **mathematical** method employed.

Answers without working may not gain full credit.

Unless the degree of accuracy is stated in the question, answers should be rounded appropriately.

You are reminded of the necessity for good English and orderly presentation in your answers.

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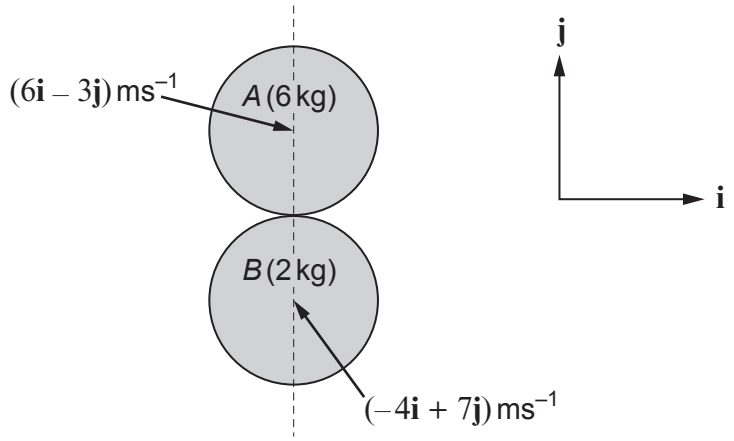
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Reminder: Sufficient working must be shown to demonstrate the **mathematical** method employed.

1. Two smooth spheres *A* and *B* are moving on a smooth horizontal plane when they collide obliquely. When the spheres collide, the line joining their centres is parallel to the vector **j**, as shown in the diagram below.

Immediately before the collision, sphere *A* has velocity $(6\mathbf{i}-3\mathbf{j})\,\text{ms}^{-1}$ and sphere *B* has velocity $(-4\mathbf{i}+7\mathbf{j})\,\text{ms}^{-1}$. Sphere *A* has mass 6 kg and sphere *B* has mass 2 kg.



Immediately after the collision, sphere *B* has velocity $(-4\mathbf{i}-5\mathbf{j})\,\text{ms}^{-1}$.

(a) Find the velocity of *A* immediately after the collision. [3]

Along line of centres (j): i-components unchanged.

Conservation of momentum in the j direction:

$6(-3) + 2(7) = 6v + 2(-5)$

$-18 + 14 = 6v - 10$

$-4 = 6v - 10 \Rightarrow v = 1$

Velocity of A after collision = $(6i + j)\ ms^{-1}$

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(b) Calculate the coefficient of restitution between A and B. [3]

Restitution along line of centres (j components):

$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{1-(-5)}{7-(-3)}$

$e = \frac{6}{10}$

$e = 0.6$

(c) Find the angle through which the direction of motion of B is deflected as a result of the collision. Give your answer correct to the nearest degree. [3]

Angle between $u_B = -4i + 7j$ and $v_B = -4i - 5j$:

$\cos\theta = \frac{u_B \cdot v_B}{|u_B|\,|v_B|} = \frac{(-4)(-4)+(7)(-5)}{\sqrt{65}\times\sqrt{41}}$

$\cos\theta = \frac{16-35}{\sqrt{65}\sqrt{41}} = \frac{-19}{\sqrt{2665}} = -0.368048\ldots$

$\theta = 111.595\ldots°$

Angle of deflection = 112° (nearest degree)

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(d) After the collision, sphere *B* continues to move with velocity $(-4\mathbf{i}-5\mathbf{j})\,\text{ms}^{-1}$ until it collides with another sphere *C*, which exerts an impulse of $(-20\mathbf{i}+18\mathbf{j})\,\text{Ns}$ on *B*.

Find the velocity of *B* after the collision with *C*. [2]

Impulse = change in momentum

$(-20i + 18j) = 2v - 2(-4i - 5j)$

$-20i + 18j = 2v + 8i + 10j$

$2v = -28i + 8j$

Velocity of B after collision = $(-14i + 4j)\ ms^{-1}$

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2. An object, of mass 1·8 kg, is falling vertically downwards under gravity. During the motion, it experiences a variable resistance of $0{\cdot}2v^2$ N, where v ms^{-1} is the speed of the object at time t seconds.

(a) Show that v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{9g - v^2}{9}.$$

[3]

N2L downwards (weight down, resistance $0.2v^2$ up):

$1.8g - 0.2v^2 = 1.8a$, where $a = \frac{dv}{dt}$

$1.8g - 0.2v^2 = 1.8\frac{dv}{dt}$ (× 5)

$9g - v^2 = 9\frac{dv}{dt}$

$\therefore \frac{dv}{dt} = \frac{9g - v^2}{9}$ as required

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At time $t = 0$, the object passes a point A with a speed of $\sqrt{g}$ ms^{-1}. The object then hits the ground with a speed of 8 ms^{-1}.

(b) Calculate the time taken for the object to hit the ground. [6]

Separating variables:

$9\int \frac{1}{9g - v^2}\, dv = \int dt$

Partial fractions, with $9g - v^2 = (3\sqrt{g} - v)(3\sqrt{g} + v)$:

$\frac{1}{9g - v^2} = \frac{1}{6\sqrt{g}}\left(\frac{1}{3\sqrt{g} - v} + \frac{1}{3\sqrt{g} + v}\right)$

$\therefore \frac{9}{6\sqrt{g}} \ln\frac{3\sqrt{g} + v}{3\sqrt{g} - v} = t + C$

i.e. $t = \frac{3}{2\sqrt{g}} \ln\frac{3\sqrt{g} + v}{3\sqrt{g} - v} - C$

When $t = 0$, $v = \sqrt{g}$:

$0 = \frac{3}{2\sqrt{g}} \ln\frac{3\sqrt{g} + \sqrt{g}}{3\sqrt{g} - \sqrt{g}} - C \Rightarrow C = \frac{3}{2\sqrt{g}} \ln 2$

$\therefore t = \frac{3}{2\sqrt{g}} \ln\frac{3\sqrt{g} + v}{3\sqrt{g} - v} - \frac{3}{2\sqrt{g}} \ln 2$

When the object hits the ground, $v = 8$:

$t = \frac{3}{2\sqrt{g}} \left[\ln\frac{3\sqrt{g} + 8}{3\sqrt{g} - 8} - \ln 2\right]$

$t = \frac{3}{2\sqrt{9.8}} \ln\frac{3\sqrt{9.8} + 8}{2(3\sqrt{9.8} - 8)}$

$t = 0.479157... \times 1.832461...$

$t = 0.878037...$

Time taken = 0.878 s (3 s.f.)

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(c) Given that the distance of the object from A at time t is x metres, form another differential equation to find an expression for x in terms of v. Hence, find the height of A above the ground. [7]

Using $a = v\frac{dv}{dx}$: $v\frac{dv}{dx} = \frac{9g - v^2}{9}$

Separating variables:

$\int \frac{9v}{9g - v^2}\, dv = \int dx$

$-\frac{9}{2}\int \frac{-2v}{9g - v^2}\, dv = \int dx$

$-\frac{9}{2}\ln|9g - v^2| = x + C$

When $x = 0$, $v = \sqrt{g}$:

$-\frac{9}{2}\ln|9g - g| = C \Rightarrow C = -\frac{9}{2}\ln(8g)$

$x = \frac{9}{2}\ln(8g) - \frac{9}{2}\ln|9g - v^2|$

$x = \frac{9}{2}\ln\frac{8g}{9g - v^2}$

The object hits the ground when $v = 8$:

$x = \frac{9}{2}\ln\frac{8(9.8)}{9(9.8) - 8^2} = \frac{9}{2}\ln\frac{78.4}{24.2}$

$x = 5.289620...$

Height of A above the ground = 5.29 m (3 s.f.)

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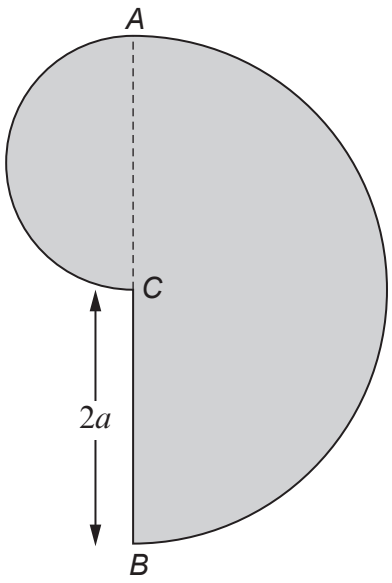
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3. *ACB* is the diameter of a semi-circular lamina of radius $2a$ and centre *C*. Another semi-circular lamina, having *AC* as its diameter, is added to form a uniform lamina, as shown in the diagram below.



(a) (i) Show that the distance of the centre of mass of the lamina from *AB* is $\frac{28}{15\pi}a$.

(ii) Calculate the distance of the centre of mass of the lamina from a line drawn through *A* that is perpendicular to *AB*. [10]

Take mass per unit area ρ. Large semicircle (radius 2a) to the right of AB, small semicircle (radius a, diameter AC) to the left.

Large: mass $\frac{1}{2}\pi(2a)^2\rho = 2\pi a^2\rho$, dist. from AB = $\frac{4(2a)}{3\pi} = \frac{8a}{3\pi}$

Small: mass $\frac{1}{2}\pi a^2\rho$, dist. from AB = $\frac{4a}{3\pi}$ (other side)

Lamina: mass $2\pi a^2\rho + \frac{1}{2}\pi a^2\rho = \frac{5}{2}\pi a^2\rho$

(i) Moments about AB (positive on the side of the large one):

$\frac{5}{2}\pi a^2\rho\,\bar{x} = 2\pi a^2\rho\,\frac{8a}{3\pi} - \frac{1}{2}\pi a^2\rho\,\frac{4a}{3\pi}$

$\frac{5}{2}\pi a^2\,\bar{x} = \frac{16a^3}{3} - \frac{2a^3}{3} = \frac{14a^3}{3}$

$\bar{x} = \frac{14a^3}{3} \times \frac{2}{5\pi a^2} = \frac{28a}{15\pi}$

∴ distance of centre of mass from AB is $\frac{28}{15\pi}a$ as required

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(ii) Distances from the line through A perpendicular to AB:

Large semicircle: centre of mass on AB at C, distance 2a from A

Small semicircle: centre of mass on AB at mid-point of AC, distance a

Moments about the line through A:

$\frac{5}{2}\pi a^2\rho\,\bar{y} = (2\pi a^2\rho)(2a) + (\frac{1}{2}\pi a^2\rho)(a)$

$\frac{5}{2}\pi a^2\,\bar{y} = 4\pi a^3 + \frac{1}{2}\pi a^3 = \frac{9}{2}\pi a^3$

$\bar{y} = \frac{9}{2}\pi a^3 \times \frac{2}{5\pi a^2} = \frac{9}{5}a$

Distance from the line through A $= \frac{9}{5}a \quad (= 1.8a)$

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(b) Suppose that the lamina is suspended in equilibrium by means of two vertical wires attached at *A* and *B* so that *AB* is horizontal. Find the fraction of the lamina's weight that is supported by the wire attached at *B*. [3]

Let the tensions be T_A at A and T_B at B, weight W.

The centre of mass is $\frac{9}{5}a$ from A along AB, and AB = 4a.

Moments about A:

$T_B \times 4a = W \times \frac{9}{5}a$

$T_B = \frac{9}{20}W$

Fraction of the weight supported by the wire at B = $\frac{9}{20}$

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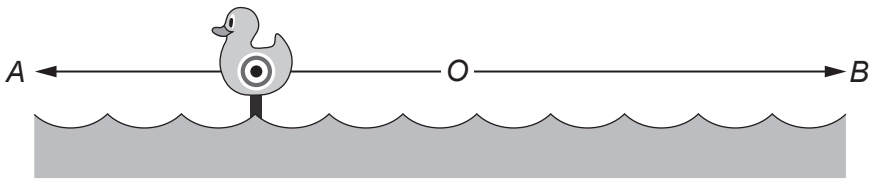
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4. The diagram below shows part of a game at a funfair that consists of a target moving along a straight horizontal line *AB*. The centre of the target may be modelled as a particle moving with Simple Harmonic Motion about centre *O*, where *O* is the midpoint of *AB*.



When the target is at a distance of 84 cm from *O*, its speed is 52 cms^{-1} and the magnitude of its acceleration is 1344 cms^{-2}.

(a) Show that the period of the motion is $\frac{\pi}{2}$ s. [3]

SHM: $\frac{d^2x}{dt^2} = -\omega^2 x$, so $|\text{acceleration}| = \omega^2|x|$

$1344 = \omega^2 \times 84$

$\omega^2 = 16 \Rightarrow \omega = 4$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{4}$

$T = \frac{\pi}{2}$ s as required

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(b) Determine the maximum speed of the target. [5]

$v^2 = \omega^2(a^2 - x^2)$ with $\omega = 4$, $x = 84$, $v = 52$ (amplitude a cm)

$52^2 = 4^2(a^2 - 84^2)$

$2704 = 16(a^2 - 7056)$

$a^2 - 7056 = 169$

$a^2 = 7225 \Rightarrow a = 85$ cm

Maximum speed $= a\omega = 85 \times 4$

Maximum speed $= 340$ cm s^{-1}

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(c) During a game, players fire a ball at the target. A timer is started when the target is at *A*. Players must wait for the target to complete at least one full cycle before firing. Given that the target is hit when it is at a distance of 67 cm from *O*, calculate the two earliest possible times taken to hit the target. [5]

Timer starts at A (an end point), so take $x = 85\cos(4t)$, where x is measured from O in the direction of A.

Target is 67 cm from O when $x = 67$ or $x = -67$.

$x = 67$: $85\cos(4t) = 67 \Rightarrow 4t = \cos^{-1}\left(\frac{67}{85}\right) = 0.662860...$

$t = 0.165715...$

$x = -67$: $85\cos(4t) = -67 \Rightarrow 4t = \cos^{-1}\left(\frac{-67}{85}\right) = 2.478732...$

$t = 0.619683...$

Must wait at least one full cycle, $T = \frac{\pi}{2} = 1.570796...$ s, so add T:

$t = \frac{\pi}{2} + 0.165715... = 1.736511...$

$t = \frac{\pi}{2} + 0.619683... = 2.190479...$

Two earliest times: 1.74 s and 2.19 s (3 s.f.)

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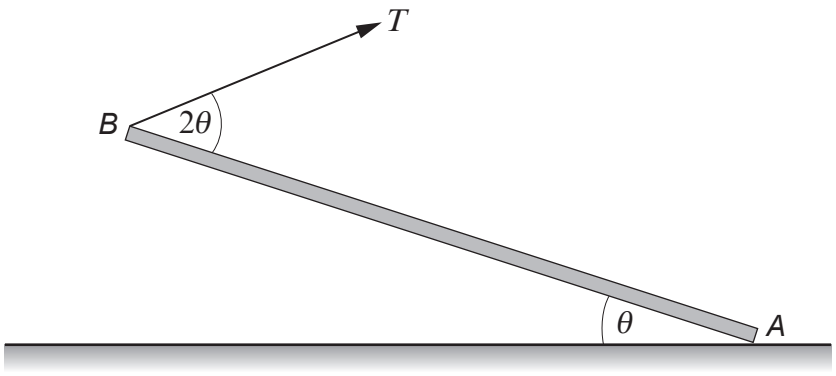
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5. The diagram below shows a uniform rod *AB* of weight WN and length $2l$, with its lower end *A* resting on a rough horizontal floor. A light cable is attached to the other end *B*. The rod is in equilibrium when it is inclined at an angle of θ to the floor, where $0^\circ < \theta \leqslant 45^\circ$. The tension in the cable is TN acting at an angle of 2θ to the rod, as shown in the diagram.



(a) (i) Show that $T = \frac{W}{4}\text{cosec}\,\theta$. [4]

(ii) Hence determine the normal reaction of the floor on the rod at *A*, giving your answer in terms of W. [4]

(i) T acts at angle 2θ to BA, so T makes angle θ with the horizontal.

Moments about A (rod length 2l, weight acts at mid-point):

$T \times 2l \sin 2\theta = W \times l \cos\theta$

$T \times 2l \times 2 \sin\theta \cos\theta = W\, l \cos\theta$

$4T \sin\theta = W \qquad (\cos\theta \neq 0)$

$T = \frac{W}{4 \sin\theta} = \frac{W}{4}\,\text{cosec}\,\theta$ as required

(ii) Resolve vertically (R = normal reaction at A):

$R + T \sin\theta = W$

$R + \frac{W}{4}\,\text{cosec}\,\theta \sin\theta = W$

$R + \frac{W}{4} = W$

$R = \frac{3}{4}W$

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(b) Given that the coefficient of friction between the floor and the rod is $\frac{\sqrt{3}}{3}$, calculate the minimum possible value for θ. [5]

Resolve horizontally (F = friction at A):

$F = T\cos\theta = \frac{W}{4}\ \text{cosec}\ \theta \cos\theta = \frac{W}{4}\cot\theta$

Equilibrium requires $F \leq \mu R$ with $\mu = \frac{\sqrt{3}}{3}$

$\frac{W}{4}\cot\theta \leq \frac{\sqrt{3}}{3} \times \frac{3}{4}W$

$\cot\theta \leq \sqrt{3}$

$\tan\theta \geq \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\theta \geq 30°$

Minimum possible value of $\theta = 30°$

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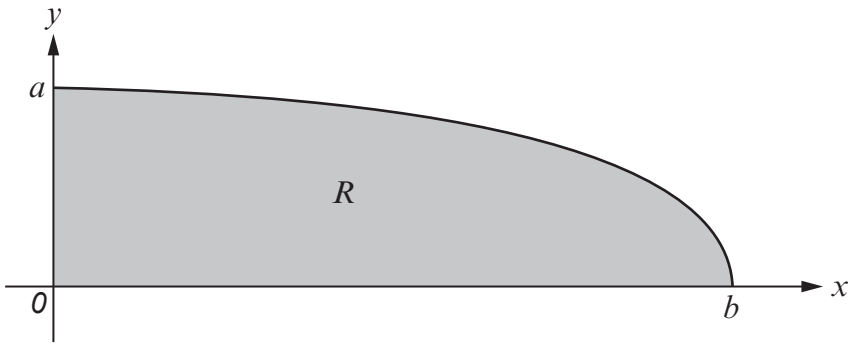
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6. The region R, shown in the diagram below, is bounded by the coordinate axes and the curve

$$y = \frac{a}{b}\sqrt{b^2 - x^2},$$

where a, b are constants.



The region R is rotated through 360° about the x-axis to form a uniform solid S.
The volume of S is $\frac{2}{3}\pi a^2 b$.

(a) Use integration to show that the distance of the centre of mass of S from the y-axis is $\frac{3b}{8}$. [5]

$V\bar{x} = \pi\int_0^b x\,y^2\,dx = \pi\int_0^b x\,\frac{a^2}{b^2}(b^2 - x^2)\,dx$

$V\bar{x} = \frac{\pi a^2}{b^2}\int_0^b (b^2x - x^3)\,dx$

$V\bar{x} = \frac{\pi a^2}{b^2}\left[\frac{b^2x^2}{2} - \frac{x^4}{4}\right]_0^b$

$V\bar{x} = \frac{\pi a^2}{b^2}\left(\frac{b^4}{2} - \frac{b^4}{4}\right) = \frac{\pi a^2}{b^2} \times \frac{b^4}{4} = \frac{1}{4}\pi a^2 b^2$

With $V = \frac{2}{3}\pi a^2 b$:

$\bar{x} = \frac{1}{4}\pi a^2 b^2 \div \frac{2}{3}\pi a^2 b = \frac{1}{4} \times \frac{3}{2}\,b$

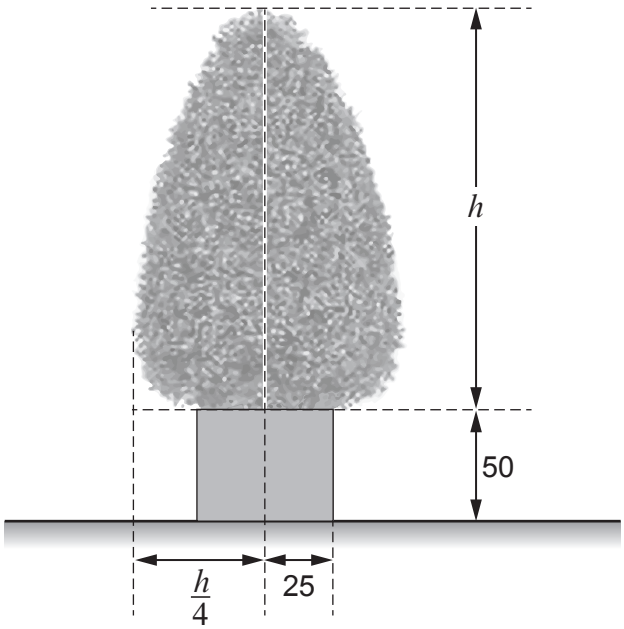
$\bar{x} = \frac{3b}{8}$ $\therefore$ distance of centre of mass from the y-axis is $\frac{3b}{8}$

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The diagram below shows a small tree growing in a pot. The uniform solid S described on the previous page may be used to model the part of the tree above the pot. This part of the tree has height h cm and base radius $\frac{h}{4}$ cm. The pot, including its contents, may be modelled as a solid cylinder of height 50 cm and radius 25 cm.



You may assume that the density of the pot, including its contents, is equal to 20 times the density of the part of the tree above the pot.

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(b) A gardener suggests that a tree is said to have outgrown its pot if the centre of mass, of both the tree and its pot, lies above the height of the pot. Determine the maximum value of h before the tree outgrows its pot. [8]

Tree: solid S with $b = h$, $a = \frac{h}{4}$ (density ρ):

mass $= \frac{2}{3}\pi(\frac{h}{4})^2 h\rho = \frac{1}{24}\pi h^3\rho$

height of c.o.m. above ground $= 50 + \frac{3h}{8}$

Pot: cylinder radius 25, height 50, density 20ρ:

mass $= \pi(25)^2(50)(20\rho) = 625000\pi\rho$

height of c.o.m. above ground $= 25$

Total mass $= \pi\rho(\frac{1}{24}h^3 + 625000)$, c.o.m. at height $\bar{h}$

Moments about the horizontal ground:

$\pi\rho[\frac{1}{24}h^3(50 + \frac{3h}{8}) + 625000 \times 25] = \pi\rho(\frac{1}{24}h^3 + 625000)\bar{h}$

Tree outgrows its pot when $\bar{h} > 50$, so the limit is $\bar{h} = 50$:

$\frac{1}{24}h^3(50 + \frac{3h}{8}) + 15625000 = 50(\frac{1}{24}h^3 + 625000)$

$\frac{50}{24}h^3 + \frac{3}{192}h^4 + 15625000 = \frac{50}{24}h^3 + 31250000$

$\frac{1}{64}h^4 = 15625000$

$h^4 = 1000000000 = 10^9$

$h = (10^9)^{1/4} = 177.827...$

Maximum value of $h = 178$ cm (3 s.f.)

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(c) Identify one possible limitation of the model used that could affect your answer to part (b). [1]

The tree is modelled as a uniform solid of exactly the shape S, but a real tree is not solid, its density is not uniform and its shape will not match the region R exactly, so the position of its centre of mass may differ.

END OF PAPER

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