Day 99 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 + 5$. Also $fg(x) = 2x^2 + 7$, where f is a linear function. (a) Find f(x). (b) Solve $gf(x) = 54$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 + 5) + b = 2x^2 + 7$.
So $a = 2$ and $10 + b = 7$, giving $b = -3$.
$f(x) = 2x - 3$.

(b) $gf(x) = (2x - 3)^2 + 5 = 54$.

$(2x - 3)^2 = 49$, so $2x - 3 = \pm 7$.

$x = -2$ or $x = 5$.

Answer: (a) $f(x) = 2x - 3$ (b) $x = -2$ or $x = 5$

2 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = x\sqrt{2x + 5}$ at the point where $x = 2$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 2$: $y = 6$. Differentiate: gradient = $\frac{11}{3}$

Tangent: $y - 6 = \frac{11}{3}(x - 2)$, so $y = \frac{11}{3}x - \frac{4}{3}$

y-intercept $-\frac{4}{3}$; x-intercept $\frac{4}{11}$

Area = $\frac{1}{2} \times |-\frac{4}{3}| \times |\frac{4}{11}|$

Answer: $\frac{8}{33}$

3 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 4x^2 + 16x + 19$.

Complete the square: $y = 4(x + 2)^2 + 3$.

Step 1: $y = x^2 \rightarrow y = 4x^2$: a stretch, scale factor 4, parallel to the y-axis.

Step 2: replace x by $(x + 2)$ and add 3: translation by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Answer: a stretch, scale factor 4, parallel to the y-axis, then translation $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4\sec^2\theta + 13\tan\theta - 1 = 0$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $4\tan^2\theta + 13\tan\theta + 3 = 0$
$(4\tan\theta + 1)(\tan\theta + 3) = 0$

$\tan\theta = -\frac{1}{4}$ or $\tan\theta = -3$

$\theta = 166.0°$ and $108.4°$ $(+180°)$

Answer: $\theta = 108.4°, 166.0°, 288.4°, 346.0°$

5 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 18 m s^{-1}. 9 s later, car B leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = 18(t + 9); B: distance = ½ × 3 × t^2
Equate: $3t^2 - 36t - 324 = 0$, so $(t - 18)(3t + 18) = 0$
t = 18 (the other root is negative)
OX = 3 × 18^2 / 2 = 486 m, and speed = 3 × 18 = 54
Answer: (a) 18 s (b) 486 m (c) 54 m s^{-1}

6 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $(n + 7)^3 - n^3 - 343$ is a multiple of 21 for every integer n.

$(n + 7)^3 = n^3 + 21n^2 + 147n + 343$.
So the expression = $21n^2 + 147n$
= 21n(n + 7), a multiple of 21.
Answer: Proved

7 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 25 pairs of values, $S_{xx} = 625$, $S_{yy} = 400$ and $S_{xy} = -318$.
(a) Calculate the product moment correlation coefficient r.
(b) The critical value for a one-tailed test at the 5% level with n = 25 is 0.3365. Carry out the test of H_0: ρ = 0 against H_1: ρ < 0.

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{-318}{\sqrt{(625 \times 400)}} = \frac{-318}{500} = -0.6360$$

H_0: ρ = 0, H_1: ρ < 0
|r| = 0.6360 > the critical value 0.3365
Reject H_0: there is sufficient evidence of negative correlation between the variables
Answer: (a) r = −0.636 (b) reject H_0

8 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 500 kg, carrying a passenger of mass 85 kg, is supported by a vertical cable. The lift is moving downwards and decelerating at 0.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is upwards
Passenger: R − 85 × 9.8 = 85 × 0.4
R = 85 × (9.8 + 0.4) = 867
Lift and passenger: T − 585 × 9.8 = (500 + 85) × 0.4
T = 585 × 10.2 = 5967
Answer: (a) 867 N (b) 5967 N

9 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 34 and variance $\frac{34}{3}$.
(a) Find n and p.
(b) Find P(X = 35), correct to 3 significant figures.

$np = 34$ and $np(1 - p) = \frac{34}{3}$
Divide: $1 - p = \frac{34}{3} \div 34 = \frac{1}{3}$, so $p = \frac{2}{3}$
$n = 34 \div \frac{2}{3} = 51$
$P(X = 35) = C(51, 35) (\frac{2}{3})^{35} (\frac{1}{3})^{16} = 0.114$
Answer: (a) n = 51, p = $\frac{2}{3}$ (b) 0.114

10 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = \ln x + 6x - 8$.

(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.

(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

$f(1) = -2.00$ and $f(2) = 4.69$: a change of sign, f continuous

$f'(x) = \frac{1}{x} + 6$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{4.693}{6.500} = 1.2780$

$x_2 = 1.2780 - \frac{-0.08686}{6.782} = 1.2908$

Answer: $x_1 = 1.2780$, $x_2 = 1.2908$