Day 97 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 10 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): ½ × 4 × 6 + ½(6 − W)(2) − W(4) = 0
12 + 6 − 1W − 4W = 0, so 5W = 18
W = 3.6

v = 0 where the line from (4, 6) to (6, −3.6) crosses the axis: $t = 4 + 2 \times \frac{6}{9.6} = 5.25$

Greatest distance = area of triangle = ½ × 5.25 × 6
Answer: (a) W = 3.6 (b) 15.75 m

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^{2x} - 10 \times e^{x} + 24 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = e^{x}$: $u^2 - 10u + 24 = 0$
$(u - 4)(u - 6) = 0$, so $e^{x} = 4$ or $e^{x} = 6$
Take logarithms: $x = \ln 4 = 1.39$; $x = \ln 6 = 1.79$
Answer: x = ln 4 = 1.39; x = ln 6 = 1.79

3 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(5r + 1) = 1650$, find n.

$\sum_{r=1}^{n}(5r + 1) = 5 \times \frac{n(n + 1)}{2} + n.$
$\frac{5\,n(n + 1)}{2} + n = 1650$, so $5n^2 + 7n - 3300 = 0$.
Solving (n > 0): n = 25.
Answer: n = 25

4 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 10 kg is at rest on rough horizontal ground (μ = 0.5). A horizontal force of 63 N acts on it for 7 s and is then removed.
Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take g = 9.8 m s^{-2}.

Friction = μmg = 0.5 × 10 × 9.8 = 49 N
While pushed: 63 − 49 = 10 a, so a = 1.4
v = 1.4 × 7 = 9.8; $s_1 = \frac{1}{2} \times 1.4 \times 7^2 = 34.3$

After release: deceleration = μg = 4.9, so $s_2 = \frac{9.8^2}{2 \times 4.9} = 9.8$

Total distance = 34.3 + 9.8
Answer: (a) 9.8 m s^{-1} (b) 44.1 m

5 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that n(n + 2)(n + 4)(n + 6) + 16 is a perfect square for every integer n.

Pair the factors: $n(n + 6) = n^2 + 6n$ and $(n + 2)(n + 4) = n^2 + 6n + 8$.
Let $u = n^2 + 6n$. Expression = u(u + 8) + 16
$= u^2 + 8u + 16 = (u + 4)^2$
$= (n^2 + 6n + 4)^2$, a perfect square.
Answer: Proved

6 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 10 cm, subtends an angle of $\frac{\pi}{2}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times \frac{\pi}{2}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 50$

Area = $25\pi - 50$

Chord AB = $2r\sin(\theta/2) = 10\sqrt{2}$

Arc AB = $r\theta = 5\pi$

Perimeter = arc + chord

Answer: Area = $-50 + 25\pi$ cm^2, perimeter = $5\pi + 10\sqrt{2}$ cm

7 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of X ~ N(76, σ^2) is shown; the shaded area is symmetrical.

(a) Find σ. (b) Find P(X > 87).

The area 0.6 leaves 0.400 in total in the tails, 0.200 in each

P(X < 71) = 0.200 so $\frac{71 - 76}{\sigma} = -0.8416$

$\sigma = \frac{5}{0.8416} = 5.941$

P(X > 87) = P(Z > 1.852) = 0.0320

Answer: (a) 5.94 (b) 0.0320

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/2} \cos^4 x \, dx$

Use $\sin^2 A$ or $\cos^2 A$ twice: write as $(\ldots)^2$ with the double angle formula

Integrate term by term

Substitute the limits

Answer: $\frac{3}{16}\pi$

9 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 - 6x + 2y - 90 = 0$

passes through P(−5, −7). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (3, −1). Gradient of radius CP = $\frac{3}{4}$.

Tangent gradient = $-\frac{4}{3}$: $-8x - 6y = 82$.

x = 0: y = $-\frac{41}{3}$. y = 0: x = $-\frac{41}{4}$.

Area = ½ × $\frac{41}{4}$ × $\frac{41}{3}$.

Answer: $\frac{1681}{24}$

10 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 7 times in round 1 and 5 times in round 2.

Given that exactly 9 wins occur altogether, find the probability that exactly 5 of them were in round 1.

Let $X \sim B(7, p)$, $Y \sim B(5, p)$ be independent; $X + Y \sim B(12, p)$

$$P(X = 5 \mid X + Y = 9) = \frac{P(X = 5)P(Y = 4)}{P(X + Y = 9)}$$

$$= \frac{C(7,5)C(5,4)\, p^9(1 - p)^3}{C(12,9)\, p^9(1 - p)^3}$$

The powers of p and $(1 - p)$ cancel, leaving $\frac{21 \times 5}{220}$

Answer: $\frac{21}{44} = 0.477$