Day 96 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\sin 6\theta \tan 3\theta}{1 - \cos 5\theta}$ approaches as $\theta \to 0$.

$\sin 6\theta \approx 6\theta$, $\tan 3\theta \approx 3\theta$, $1 - \cos 5\theta \approx \frac{25}{2}\theta^2$

Expression $\approx \frac{6\theta \times 3\theta}{\frac{25}{2}\theta^2} = \frac{36}{25}$

Answer: $\frac{36}{25}$

2 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=3}^{13}(4 \times 2^r + 1r)$.

Geometric part: first term $4 \times 2^3 = 32$, ratio 2, 11 terms.

Sum = $\frac{32(2^{11} - 1)}{1} = 65504$.

Arithmetic part: 1×11 terms, average $(3 + 13)/2$: sum = 88.

Total = 65504 + 88.

Answer: 65592

3 Normal approximation to a Poisson (CIE)

The number of customers arriving at a shop in one hour is Po(45).

Use a suitable approximation to find $P(X \leq 38)$, to 3 significant figures.

$\lambda = 45$ is large (greater than 15), so X is approximately N(45, 45)

Standard deviation = $\sqrt{45} = 6.708$

Continuity correction:

$P(X \leq 38) \approx P(Y < 38.5) = P(Z < -0.969) = 0.166$

Answer: 0.166

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 3x + 12$ has a root α.

(a) Show that the equation can be written as $x = \sqrt[3]{(3x + 12)}$.

(b) Use $x_{n+1} = \sqrt[3]{(3x_n + 12)}$ with $x_0 = 2$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(3x + 12)}$

$x_1 = 2.6207$, $x_2 = 2.7082$, $x_3 = 2.7200$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.722$

Answer: $x_1 = 2.6207$, $x_2 = 2.7082$, $x_3 = 2.7200$; $\alpha = 2.722$

5 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 3 kg lies on a smooth horizontal table, with a block of mass 3 kg on it ($\mu = 0.25$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 36.75. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{6}$. The block is accelerated only by friction: $F = 3a \leq \mu mg$

So $a \leq \mu g = 2.45$, and P_max $= 6 \times 2.45 = 14.7$

$P = 36.75 > 14.7$, so the block slips. Block: $a = \mu g = 2.45$

Plank: $36.75 - 7.35 = 3a$, so $a = 9.8$

Answer: (a) 14.7 N (b) block 2.45 m s^{-2}, plank 9.8 m s^{-2}

6 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(9, 2) lies on the circle: $(9 - r)^2 + (2 - r)^2 = r^2$.

$r^2 - 22r + 85 = 0$, so r = 5 or r = 17.

The diagram shows P on the larger circle, so r = 17.

$(x - 17)^2 + (y - 17)^2 = 289$, then expand.

Answer: $x^2 + y^2 - 34x - 34y + 289 = 0$

7 Sinusoidal velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where $v = 10\sin(\frac{\pi t}{2})$ for $t \geq 0$.

Find (a) the greatest magnitude of the acceleration of P, (b) the distance P travels between the first two times at which it is at rest. Give (b) to 3 s.f.

$a = \frac{dv}{dt} = 5\pi\cos(\frac{\pi t}{2})$, so the greatest magnitude is 5π

v = 0 when $\sin(\frac{\pi t}{2}) = 0$: t = 0 and t = 2

s = ∫ from 0 to 2 of $10\sin(\frac{\pi t}{2})\,dt = [-\frac{10 \times 2}{\pi}\cos(\frac{\pi t}{2})]$ from 0 to 2

$= \frac{20}{\pi}(1 - (-1)) = \frac{40}{\pi}$

Answer: (a) 5π m s^{-2} (b) 12.7 m

8 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2$, $y = 3t(2 - t)$,
for $0 \leq t \leq 2$. The curve meets the x-axis at O and at A.
Find the area enclosed between the curve and the x-axis.

dx/dt = 6t; x from 0 to 12 as t goes from 0 to 2

$\text{Area} = \int_0^2 y\frac{dx}{dt}\,dt = \int_0^2 18t^2(2 - t)\,dt$

$= 18[\frac{2t^3}{3} - \frac{t^4}{4}]$

Answer: 24

9 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(27 - x)^{\frac{1}{3}}$ up to and including the term in x^2.

(b) Use x = 0.03 to estimate $\sqrt[3]{26.97}$ to 5 decimal places.

(a) $(27 - x)^{\frac{1}{3}} = 3(1 - \frac{1}{27}x)^{\frac{1}{3}}$.

$= 3 - \frac{1}{27}x - \frac{1}{2187}x^2 + ...$

(b) 27 − x with x = 0.03 gives 26.97.

Substitute: $3 + (-\frac{1}{27})(0.03) + (-\frac{1}{2187})(0.03)^2$.

= 2.99889 (5 d.p.).

Answer: (a) $3 - \frac{1}{27}x - \frac{1}{2187}x^2$ (b) 2.99889

10 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(2) and Y ~ Po(3).

(a) State the distribution of X + Y.

(b) Given that X + Y = 5, find the probability that X = 4.

$X + Y \sim Po(5)$

$P(X = 4 \mid X + Y = 5) = \frac{P(X = 4)P(Y = 1)}{P(X + Y = 5)}$

$= \frac{e^{-2}2^4}{4! \times e^{-3}3^1/1!} \div \frac{e^{-5}5^5}{5!}$

$= C(5, 4)\ (\frac{2}{5})^4\ (\frac{3}{5})^1$, a binomial probability

$= 0.0768$

Answer: (a) Po(5) (b) 0.0768