Day 93 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^1 xe^{3x}\,dx$

Let u = x and dv/dx = e^{3x}

Integral = [uv] − $\int$ v du/dx dx

Evaluate between the limits

Answer: $\frac{1}{9} + \frac{2}{9}e^3$

2 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both

$x^2 - 11x + 30 \geq 0$ and $-2x - 2 \leq 2$.

Quadratic: $(x - 5)(x - 6) \geq 0$ gives $x \leq 5$ or $x \geq 6$.
Linear: $-2x - 2 \leq 2$ gives $x \geq -2$. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.
Answer: $\mathbf{-2 \leq x \leq 5}$ **or** $\mathbf{x \geq 6}$

3 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school says that 30% of its pupils walk to school. 15 pupils are surveyed, and 5 walk.
Test, at the 10% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.3, H_1: p > 0.3. Let X ~ B(15, 0.3)
P(X ≥ 5) = 1 − P(X ≤ 4) = 0.485
0.485 is greater than 10%, so do not reject H_0
There is insufficient evidence that the proportion is higher than the claimed 0.3
Answer: Do not reject H_0: p-value 0.485 > 10%

4 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + 3n + 13$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.
For n = 9: $9^2 + 39 + 13 = 121$.
121 = 11 × 11, which is not prime.
All smaller values of n give a prime, so n = 9 is the smallest.
Answer: n = 9 (value 121 = 11 × 11)

5 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of X ~ N(96, σ^2) is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find P(X > 112).

The area 0.7 leaves 0.300 in total in the tails, 0.150 in each

P(X < 91) = 0.150 so $\frac{91 - 96}{\sigma} = -1.0364$

$\sigma = \frac{5}{1.0364} = 4.824$

P(X > 112) = P(Z > 3.317) = 0.000456
Answer: (a) 4.82 (b) 0.000456

6 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 2$ to find the exact value of

$\int_0^1 x(x^2 + 2)^2 \, dx$

$du = 2x\,dx$; limits $x = 0, 1$ become $u = 2, 3$

Integral $= \frac{1}{2}\int_2^3 u^2 \, du = [\frac{u^3}{6}]$

Evaluate the limits

Answer: $\frac{19}{6}$

7 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 15.2 m, at a horizontal distance of 25 m from O.

Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$15.2 = 25 - \frac{9.8 \times 25^2}{u^2}$, so $u^2 = \frac{9.8 \times 625}{9.8} = 625$

$u = 25$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{625}{39.2}$

Answer: (a) 25 m s^{-1} (b) 15.9 m

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{6 + 3\sin\theta + 4\cos\theta}$

(a) Write $3\sin\theta + 4\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{3^2 + 4^2} = 5$, $\tan\alpha = \frac{4}{3}$, $\alpha = 53.1°$

Denominator $= 6 + 5\sin(\theta + 53.1°)$, which lies between 1 and 11

Greatest $f(\theta) = \frac{4}{1} = 4$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{11} = \frac{4}{11}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 53.1° = 270°$, so $\theta = 216.9°$

Answer: (a) $5\sin(\theta + 53.1°)$ (b) greatest 4, least $\frac{4}{11}$, $\theta = 216.9°$

9 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations

$x = 3t$, $y = t^2$. The line $y = -\frac{1}{3}x + 12$

meets C at A and B. Find the exact length AB.

Substitute: $t^2 = -t + 12$.

$t^2 + t - 12 = 0 \Rightarrow t = 3$ or $t = -4$.

A(9, 9), B(−12, 16).

$AB^2 = 21^2 + (-7)^2 = 490$.

Answer: AB $= 7\sqrt{10}$

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(8\mathbf{i} + 28\mathbf{j})$ km relative to a port O, and moves with constant velocity $(6\mathbf{i} - \mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 10 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(8\mathbf{i} + 28\mathbf{j}) + t(6\mathbf{i} - \mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 10 t, so $|(8\mathbf{i} + 28\mathbf{j}) + t(6\mathbf{i} - \mathbf{j})| = 10\,t$
Squaring and rearranging: $-63t^2 + 40t + 848 = 0$, which gives t = 4 or a negative value
t = 4 (the other root is negative since B is faster than A)

A is then at $(8\mathbf{i} + 28\mathbf{j}) + 4(6\mathbf{i} - \mathbf{j}) = (32\mathbf{i} + 24\mathbf{j})$, so $v_B = \frac{1}{4}(32\mathbf{i} + 24\mathbf{j}) = (8\mathbf{i} + 6\mathbf{j})$

Answer: (a) 4 h (b) $(8\mathbf{i} + 6\mathbf{j})$ km h^{-1}