Day 92 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A disease is present in 3% of a population. A test is positive for 90% of those with it and for 10% of those without it. Find P(positive) and P(has it | positive).

P(positive) = 0.03 × 0.9 + 0.97 × 0.1
= 0.027 + 0.097 = 0.124

P(has it | positive) = $\frac{0.027}{0.124}$

= 0.2177

Answer: (a) 0.124 (b) 0.218

2 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta + 5\sin\theta - 3 = 0$
for $0 \le \theta < 2\pi$, giving answers in terms of π.

Use $\cos 2\theta \equiv 1 - 2\sin^2\theta$: $2\sin^2\theta - 5\sin\theta + 2 = 0$
$(2\sin\theta - 1)(\sin\theta - 2) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = 2$ (impossible, $|\sin\theta| \le 1$)

Principal values and symmetry in $[0, 2\pi)$

Answer: $\boldsymbol{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$

3 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^3 + 14n$ is divisible by 3
for every positive integer n.

$n^3 + 14n = (n - 1)n(n + 1) + 15n$.
One of three consecutive integers is a multiple of 3,
so $(n - 1)n(n + 1)$ is divisible by 3.
15 is a multiple of 3, so 15n is too. Hence the sum is divisible by 3.

Answer: Proved

4 Poisson approximation to a binomial (CIE)

X ~ B(400, 0.005).
(a) Explain why a Poisson approximation is suitable and state its distribution.
(b) Use it to estimate P(X > 2), to 4 s.f.
(c) Find the exact value of P(X > 2), to 4 s.f.

n = 400 is large and p = 0.005 is small, with np = 2
X is approximately Po(2)
$P(X > 2) \approx 1 - P(Y \le 2) = 0.3233$
Exact: $1 - P(X \le 2)$ with X ~ B(400, 0.005) = 0.3233

Answer: (a) Po(2) (b) 0.3233 (c) 0.3233

5 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 10 kg is at rest on rough horizontal ground ($\mu = 0.2$). A horizontal force of 33.6 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8\ \text{m s}^{-2}$.

Friction = $\mu mg = 0.2 \times 10 \times 9.8 = 19.6$ N
While pushed: $33.6 - 19.6 = 10a$, so $a = 1.4$
$v = 1.4 \times 7 = 9.8$; $s_1 = \frac{1}{2} \times 1.4 \times 7^2 = 34.3$

After release: deceleration = $\mu g = 1.96$, so $s_2 = \frac{9.8^2}{2 \times 1.96} = 24.5$

Total distance = 34.3 + 24.5

Answer: (a) 9.8 m s^{-1} (b) 58.8 m

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6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2 \sin t$, $y = 4 \cos 2t$.
Find a cartesian equation of the curve in the form $y = f(x)$.

$\cos 2t = 1 - 2\sin^2 t$

$\sin t = \frac{x}{2}$, so $y = 4(1 - 2 \times \frac{x^2}{4})$

Answer: $y = 4 - 2x^2$

7 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 10 \times 2^x + 16 = 0$.

Let $y = 2^x$. Then $y^2 - 10y + 16 = 0$.
$(y - 2)(y - 8) = 0$, so $y = 2$ or $y = 8$.
$2^x = 2$ gives $x = 1$; $2^x = 8$ gives $x = 3$.
Answer: $x = 1$ or $x = 3$

8 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 3xy + 5y^2 = 11$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -3y}{-3x + 10y}$

Horizontal tangent when $2x + -3y = 0$, so $y = \frac{2x}{3}$

Substitute into the equation of C: $x^2 + (-3xy) + (5y^2) = 11$
Solve for the remaining coordinate
Answer: $(-3, -2)$ and $(3, 2)$

9 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = \frac{1}{3}$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find $\sum_{r=1}^{60} u_r$.

$u_1 = \frac{1}{3}$, $u_2 = \frac{3}{2}$, $u_3 = -2$, $u_4 = \frac{1}{3}$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $-\frac{1}{6}$.

$60 = 3 \times 20 + 0$: 20 full periods plus the first 0 terms.

Answer: $-\frac{10}{3}$

10 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 45 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{7}{24}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{4}{3}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 45\cos\alpha$, so $R = 45 \times \frac{24}{25} - P\sin\beta$

Along the plane: $P\cos\beta = 45\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 45(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5}) = 45(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25})$, so $P = 37.5$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_min = 45(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25}) \div \frac{5}{4}$

Answer: (a) 37.5 N (b) 36 N