Day 90 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{76 - 42\sqrt{3}}$ in the form
$a - b\sqrt{3}$, where a and b are positive integers.

Suppose $\sqrt{76 - 42\sqrt{3}} = a - b\sqrt{3}$. Square both sides.
$76 - 42\sqrt{3} = a^2 + 3b^2 - 2ab\sqrt{3}$.
Equate: $a^2 + 3b^2 = 76$ and $2ab = 42$, so $ab = 21$.
Try $a = 7$, $b = 3$: $49 + 27 = 76$ □.
($a > b\sqrt{3}$, so the root is positive.)
Answer: $7 - 3\sqrt{3}$

2 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(5\mathbf{i} + 7\mathbf{j})$ N, $(4\mathbf{i} + k\mathbf{j})$ N and $(8\mathbf{i} - \mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(17\mathbf{i} - \mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(17)\mathbf{i} + (6 + k)\mathbf{j}$
Parallel to $(17\mathbf{i} - \mathbf{j})$: $\frac{6 + k}{17} = \frac{-1}{17}$, so $k = -7$
$R = (17\mathbf{i} - \mathbf{j})$ N, so $|R| = \sqrt{(17^2 + -1^2)} = 17.03$
$a = |R| \div 4$
Answer: (a) $k = -7$ (b) 4.26 m s^{-2}

3 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at $x = 1$ and touches at $x = -3$, so $f(x) = k(x - 1)(x + 3)^2$.
At (0, −18): $-18 = k(-1)(3)^2 = -9k$, so $k = 2$.
$f(x) = 2(x - 1)(x + 3)^2$, then expand.
Answer: $f(x) = 2x^3 + 10x^2 + 6x - 18$

4 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

12 recorded masses (grams) have mean 59 and standard deviation 5.
One value was recorded as 52 but should have been 63.
Find the correct mean and the correct standard deviation.

$\Sigma x = 12 \times 59 = 708$, so corrected $\Sigma x = 708 - 52 + 63 = 719$
Correct mean $= \frac{719}{12} = 59.92$
$\Sigma x^2 = 12(25 + 3481) = 42072$, corrected $\Sigma x^2 = 42072 - 2704 + 3969 = 43337$
Variance $= \frac{43337}{12} - 59.917^2 = 21.410$, $\sigma = 4.627$
Answer: mean 59.9, standard deviation 4.63

5 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = 5x - 10 \sin x$ for $0 \leq x \leq 2\pi$,
and determine their nature.

$dy/dx = 5 - 10 \cos x = 0$, so $\cos x = \frac{1}{2}$
$x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$
$d^2y/dx^2 = 10 \sin x$
Its sign at each point decides maximum or minimum
Answer: $\left(\frac{\pi}{3}, \frac{5}{3}\pi - 5\sqrt{3}\right)$ minimum; $\left(\frac{5\pi}{3}, \frac{25}{3}\pi + 5\sqrt{3}\right)$ maximum

6 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(5r - 2) = 731$, find n.

$\sum_{r=1}^{n}(5r - 2) = 5 \times \frac{n(n + 1)}{2} - 2n.$

$\frac{5\,n(n + 1)}{2} - 2n = 731$, so $5n^2 + n - 1462 = 0$.

Solving $(n > 0)$: $n = 17$.

Answer: n = 17

7 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.

Initially P = 120 and after 4 hours P = 300.

Find k, and the time for the population to reach 960, to 3 s.f.

$P_0 = 120$. $120\,e^{4k} = 300$ so $e^{4k} = \frac{5}{2}$

$k = \frac{1}{4}\ln\frac{5}{2} = 0.229$

$120\,e^{kt} = 960$ so $e^{kt} = 8$, $t = \frac{1}{k}\ln 8$

t = 9.08 hours

Answer: k = 0.229, t = 9.08 hours

8 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 10 components, each faulty with probability 0.1, independently. A box is rejected if it has at least 3 faulty components.

(a) Find the probability that a box is rejected.

(b) 15 boxes are inspected. Find the probability that at least 2 are rejected.

Let $X \sim B(10, 0.1)$. $P(\text{rejected}) = P(X \geq 3) = 1 - P(X \leq 2) = 0.07019$

Let Y = number of rejected boxes, $Y \sim B(15, 0.07019)$

$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.284$

Answer: (a) 0.0702 (b) 0.284

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{3x}\cos 2x$. Show that $dy/dx = e^{3x}(3\cos 2x - 2\sin 2x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (3)e^{3x}\cos 2x - e^{3x} \times 2\sin 2x$

$dy/dx = 0$ requires $3\cos 2x = 2\sin 2x$

$\tan 2x = \frac{3}{2}$, smallest positive $2x = 0.9828$

x = 0.491

Second derivative test (or gradient change): a maximum

Answer: x = 0.491, maximum

10 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 8 m and weight 100 N has A on rough ground ($\mu = 0.4$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = 2$.
A man of weight 500 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2} \cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, R = W: need $\frac{W}{2\tan\theta} \leq \mu W$, so $\mu \geq \frac{1}{2 \times 2} = 0.25$

With the man at distance x: R = 600, and moments about A give $S L \sin\theta = (100 \times 4 + 500x)\cos\theta$

Limiting: $S = \mu R = 0.4 \times 600 = 240$, so $240 \times 8 \times 2 = 100 \times 4 + 500x$

$500x = 3840 - 400$, so $x = 6.88$

Answer: 6.88 m