Day 9 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - x^2 + 5x - 14$.
Show that $(x - 2)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(2) = 8 - 4 + 10 - 14 = 0$, so $(x - 2)$ is a factor.

$f(x) = (x - 2)(x^2 + x + 7)$.

For $x^2 + x + 7 = 0$: $b^2 - 4ac = 1 - 28 = -27 < 0$.

No real roots from the quadratic, so the only real root is $x = 2$.

Answer: Proved; the only real root is $x = 2$

2 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The times taken by a bus, in minutes are normally distributed. 10% are less than 56 and 40% are greater than 74.
Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 56) = 0.10$ gives $\frac{56 - \mu}{\sigma} = -1.2816$

$P(X > 74) = 0.40$ gives $\frac{74 - \mu}{\sigma} = 0.2533$

Subtract: $\frac{18}{\sigma} = 0.2533 - (-1.2816) = 1.5349$

$\sigma = 11.73$

$\mu = 56 - (-1.2816)(11.73) = 71.03$

Answer: $\mu = 71.0$, $\sigma = 11.7$

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Find (a) the time of flight, (b) the range, (c) the greatest height. Take $g = 9.8$ m s^{-2}.

Components: horizontal 14.7, vertical 19.6 (as $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$)

Time of flight: $0 = 19.6\,t - 4.9t^2$, so $t = \frac{19.6}{4.9} = 4$

Range $= 14.7 \times 4 = 58.8$

Greatest height: $0 = 19.6^2 - 2 \times 9.8 \times H$, so $H = \frac{384.16}{19.6} = 19.6$

Answer: (a) 4 s (b) 58.8 m (c) 19.6 m

4 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 3x^2 - 24x + 49$ for $x \geq k$. Given that k is the least value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 3(x - 4)^2 + 1$.

The vertex is at $x = 4$, so $k = 4$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 1 = 3(x - 4)^2$, so $(x - 4) = \pm\sqrt{\frac{y - 1}{3}}$.

As $x \geq 4$, take the positive root: $x = 4 + \sqrt{\frac{y - 1}{3}}$.

Range of f = domain of f^{-1}: $f(4) = 1$, so $x \geq 1$.

Answer: $k = 4$; $f^{-1}(x) = 4 + \sqrt{\frac{x - 1}{3}}$; $x \geq 1$

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 18 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 90$, and after 5 minutes $\theta = 72$.
Find A and k, then find when $\theta = 22$ (3 s.f.).

$t = 0$: $18 + A = 90$ so $A = 72$

$t = 5$: $18 + 72e^{-5k} = 72$ so $e^{-5k} = \frac{3}{4}$

$k = \frac{1}{5} \ln \frac{4}{3} = 0.0575$

$72\, e^{-kt} = 4$ so $t = \frac{1}{k} \ln 18$

$t = 50.2$ minutes

Answer: A = 72, k = 0.0575, t = 50.2 minutes

6 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The breaking strength of a cable is $N(\mu, 25)$ with known variance 25. The mean was claimed to be 55 kN. A random sample of 36 has mean 56.5. Test at the 10% level whether the mean is less than 55.

$H_0: \mu = 55$, $H_1: \mu < 55$. Under H_0, the sample mean ~ $N(55, \frac{25}{36})$

$z = \frac{56.5 - 55}{5 \div \sqrt{36}} = 1.800$

p-value = $P(Z < 1.800) = 0.964$

$0.964 > 10\%$, so do not reject H_0

There is insufficient evidence at the 10% level that the mean has changed as claimed

Answer: Do not reject H_0 (z = 1.800, p-value 0.964)

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates $(-2, 3, 2)$, $(3, 1, 2)$, $(2, -3, -1)$.
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = -3$

$|\mathrm{BA}| = \sqrt{29}$, $|\mathrm{BC}| = \sqrt{26}$

$\cos \mathrm{ABC} = \frac{-3}{\sqrt{29}\sqrt{26}} = -0.1093$, so ABC = 96.3°

$|\mathrm{BA} \times \mathrm{BC}| = \left|\begin{pmatrix} -6 \\ -15 \\ 22 \end{pmatrix}\right| = \sqrt{745}$

Area $= \frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 96.3°, area = $\frac{1}{2}\sqrt{745}$

8 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix}3\\2\\4\end{pmatrix} + t\begin{pmatrix}2\\2\\2\end{pmatrix}$.

The point P has coordinates $(0, -6, 3)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix}-3\\-8\\-1\end{pmatrix} - t\begin{pmatrix}2\\2\\2\end{pmatrix}$

FP · d = 0: $-24 - t(12) = 0$, so $t = -2$

F = $(-1, -2, 0)$

$\mathrm{PF} = \begin{pmatrix}1\\-4\\3\end{pmatrix}$, so $\mathrm{PF} = \sqrt{26}$

Answer: F = $(-1, -2, 0)$, PF = $\sqrt{26}$

9 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=6}^{11}(5 \times 3^r + 4r)$.

Geometric part: first term $5 \times 3^6 = 3645$, ratio 3, 6 terms.

Sum = $\frac{3645(3^6 - 1)}{2} = 1326780$.

Arithmetic part: 4 × 6 terms, average (6 + 11)/2: sum = 204.

Total = 1326780 + 204.

Answer: 1326984

10 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 8 m s^{-1}. 6 s later, car B leaves O from rest with constant acceleration 2 m s^{-2}, travelling in the same direction.

B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = 8(t + 6); B: distance = ½ × 2 × t^2

Equate: $2t^2 - 16t - 96 = 0$, so $(t - 12)(2t + 8) = 0$

t = 12 (the other root is negative)

OX = $2 \times 12^2 / 2 = 144$ m, and speed = $2 \times 12 = 24$

Answer: (a) 12 s (b) 144 m (c) 24 m s^{-1}