Day 89 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 4 m s^{-1} up a line of greatest slope of a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$ and $\mu = 0.4$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take g = 9.8 m s^{-2}.

$\cos\alpha = \frac{3}{5}$. Up the plane: $R = mg\cos\alpha$, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{4}{5} + 0.4 \times \frac{3}{5}) = 10.192$

$0 = 4^2 - 2 \times 10.192 \times d$, so $d = \frac{16}{20.384} = 0.7849$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{4}{5} - 0.4 \times \frac{3}{5}) = 5.488$

$v^2 = 2 \times 5.488 \times 0.78493 = 8.615$, so $v = 2.94$

Answer: (a) 0.785 m (b) 2.94 m s^{-1}

2 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{49 + 20\sqrt{6}}$ in the form
$a + b\sqrt{6}$, where a and b are positive integers.

Suppose $\sqrt{49 + 20\sqrt{6}} = a + b\sqrt{6}$. Square both sides.
$49 + 20\sqrt{6} = a^2 + 6b^2 + 2ab\sqrt{6}$.
Equate: $a^2 + 6b^2 = 49$ and $2ab = 20$, so $ab = 10$.
Try a = 5, b = 2: 25 + 24 = 49 ☐.
($a > b\sqrt{6}$, so the root is positive.)

Answer: $5 + 2\sqrt{6}$

3 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{3}{5}$. A force P N acts up a line of greatest slope; the particle stays at rest for $30 \le P \le 150$.

Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 300. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 150 and up the plane at P = 30
$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$

Adding: $2W\sin\alpha = 180$, so $W = \frac{180}{2 \times \frac{3}{5}} = 150$

Subtracting: $2F = 120$, so $F = 60 = \mu \times 150 \times \frac{4}{5}$, giving $\mu = 0.5$

P = 300: net force up the plane = 300 − 150 = 150, and mass = $\frac{150}{9.8}$

Answer: (a) 150 N (b) 0.5 (c) 9.8 m s^{-2}

4 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_3(x + 27) + \log_3(x - 53) = 4$,
justifying clearly any root that you reject.

$(x + 27)(x - 53) = 3^4 = 81$
$x^2 - 26x - 1512 = 0$
$(x - 54)(x + 28) = 0$, so $x = 54$ or $x = -28$
$x = -28$ makes $x + 27 = -1 < 0$ and log of a negative number is undefined
Answer: x = 54

5 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 4$ and $v = \frac{y}{20}$. The regression line of v on u is $v = 3.8 + 6u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 6$.

Substitute $u = x - 4$ and $v = y/20$: $\frac{y}{20} = 3.8 + 6(x - 4)$

$y = 20[3.8 + 6x - 24] = 20 \times -20.2 + 20 \times 6x$
$y = -404 + 120x$
At $x = 6$: $y = -404 + 120 \times 6 = 316$
Answer: (a) $y = -404 + 120x$ (b) 316

6 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of

$y = -2f(-\frac{1}{2}x + 1)$.

Inside f: replace x by $-\frac{1}{2}x + 1$, so x-coordinates become $(x - 1) \div (-\frac{1}{2})$.

Outside: y-coordinates become $-2 \times y$.
A: (14, 2) B: (10, −2)
C: (6, 6) D: (−2, 0)
Answer: A′(14, 2), B′(10, −2), C′(6, 6), D′(−2, 0)

7 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{8}$ and $\tan B = \frac{7}{9}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{8} + \frac{7}{9}}{1 - \frac{7}{72}} = \frac{\frac{65}{72}}{\frac{65}{72}} = 1$$

A + B is acute, so $A + B = \pi/4$

Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$

8 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bulb's lifetime, in hours, is $N(58, 12^2)$.
Given that it exceeds 46, find the probability that it exceeds 60.

Required $= P(X > 60 \mid X > 46) = \frac{P(X > 60)}{P(X > 46)}$
$P(X > 60) = P(Z > 0.167) = 0.4338$
$P(X > 46) = P(Z > -1.000) = 0.8413$
$= \frac{0.4338}{0.8413} = 0.516$
Answer: 0.516

9 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_3 5$ is irrational.

Assume $\log_3 5 = \frac{m}{n}$, m, n positive integers.

Then $3^{\frac{m}{n}} = 5$, so $3^m = 5^n$.
Prime factors of 3^m and 5^n must be identical,
but 5 divides 5^n but 5 does not divide 3^m.
This is a contradiction, so $\log_3 5$ is irrational.
Answer: Proved

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10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = y(3 - y)$, and $y = \frac{3}{2}$ when $x = 0$.

Solve to find the value of x when $y = \frac{9}{4}$.

$\frac{1}{y(3-y)} \equiv \frac{1}{3}(\frac{1}{y} + \frac{1}{3-y})$

$\frac{1}{3}\ln|\frac{y}{3-y}| = x + c$; $x = 0$, $y = \frac{3}{2}$ gives $c = 0$

$\frac{y}{3-y} = e^{3x}$; with $y = \frac{9}{4}$: $3 = e^{3x}$

Answer: $x = \frac{\ln 3}{3}$