Day 87 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 1) + \log_2(x - 13) = 5$,
justifying clearly any root that you reject.

$(x + 1)(x - 13) = 2^5 = 32$
$x^2 - 12x - 45 = 0$
$(x - 15)(x + 3) = 0$, so $x = 15$ or $x = -3$
$x = -3$ makes $x + 1 = -2 < 0$ and log of a negative number is undefined
Answer: $x = 15$

2 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 20 and upper quartile 36. An outlier is a value more than 1 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 8 observations have mean 29. The largest is 57, an outlier, and it is removed. Find the new mean.

IQR = 36 − 20 = 16, so 1 × IQR = 16
Lower limit = 20 − 16 = 4, upper limit = 36 + 16 = 52
$\Sigma x = 8 \times 29 = 232$, so without 57: $\Sigma x = 175$
New mean = $\frac{175}{7}$
Answer: (a) 4 to 52 (b) 25.00

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-6 - 2\sqrt{2}}{1 - 2\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(1 + 2\sqrt{2})$.
Numerator: $(-6 - 2\sqrt{2})(1 + 2\sqrt{2}) = -14 - 14\sqrt{2}$.
Denominator: $1^2 - 2^2(2) = -7$.
So the fraction = $\frac{-14 - 14\sqrt{2}}{-7}$.
Answer: $2 + 2\sqrt{2}$

4 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 4$ and $v = \frac{y}{5}$. The regression line of v on u is $v = 5.8 + 0.1u$.
(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 12$.

Substitute $u = x - 4$ and $v = y/5$: $\frac{y}{5} = 5.8 + 0.1(x - 4)$
$y = 5[5.8 + 0.1x - 0.4] = 5 \times 5.4 + 5 \times 0.1x$
$y = 27 + 0.5x$
At $x = 12$: $y = 27 + 0.5 \times 12 = 33$
Answer: (a) $y = 27 + 0.5x$ (b) 33

5 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = \ln t$, $y = 3t^2 + 3t$, $t > 0$.
Find the equation of the tangent where $t = 1$, in the form $y = mx + c$.

$dx/dt = \frac{1}{t}$, $dy/dt = 6t + 3$
$dy/dx = t(6t + 3)$ which is 9 at $t = 1$
At $t = 1$: $x = 0$, $y = 6$
Answer: $y = 9x + 6$

6 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(-2\mathbf{i} + 2\mathbf{j})$ m with velocity $(9\mathbf{i} + 7\mathbf{j})$ m s^{-1}. At t = 6, P has velocity $(-15\mathbf{i} - 17\mathbf{j})$ m s^{-1}.

Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{i} - \mathbf{j})$, (c) the position vector of P at that time.

$a = \frac{v - u}{t} = \frac{(-15\mathbf{i} - 17\mathbf{j}) - (9\mathbf{i} + 7\mathbf{j})}{6} = (-4\mathbf{i} - 4\mathbf{j})$

$v = (9\mathbf{i} + 7\mathbf{j}) + t(-4\mathbf{i} - 4\mathbf{j})$

Parallel to $(\mathbf{i} - \mathbf{j})$: $(-4t + 9) \times -1 = (-4t + 7) \times 1$, giving t = 2

$r = (-2\mathbf{i} + 2\mathbf{j}) + 2(9\mathbf{i} + 7\mathbf{j}) + \frac{1}{2} \times 2^2 (-4\mathbf{i} - 4\mathbf{j}) = (8\mathbf{i} + 8\mathbf{j})$

Answer: (a) $(-4\mathbf{i} - 4\mathbf{j})$ m s^{-2} (b) t = 2 (c) $(8\mathbf{i} + 8\mathbf{j})$ m

7 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 3}{x - 5}$ for $x \neq 5$. Given that f is self-inverse, find k, and hence solve f(x) = x.

Let y = f(x): $y(x - 5) = kx + 3$, so $x(y - k) = 5y + 3$.

$f^{-1}(x) = \frac{5x + 3}{x - k}$.

$f = f^{-1}$ requires k = 5.

f(x) = x: $5x + 3 = x(x - 5)$, so $x^2 - 10x - 3 = 0$.

$x = 5 \pm \sqrt{28} = 5 \pm 2\sqrt{7}$.

Answer: k = 5; $x = 5 \pm 2\sqrt{7}$

8 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(25 - x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use x = 0.05 to estimate $\frac{1}{\sqrt{24.95}}$ to 5 decimal places.

(a) $(25 - x)^{-\frac{1}{2}} = \frac{1}{5}(1 - \frac{1}{25}x)^{-\frac{1}{2}}$.

$= \frac{1}{5} + \frac{1}{250}x + \frac{3}{25000}x^2 + ...$

(b) 25 − x with x = 0.05 gives 24.95.

Substitute: $\frac{1}{5} + \frac{1}{250}(0.05) + \frac{3}{25000}(0.05)^2$.

= 0.20020 (5 d.p.).

Answer: (a) $\frac{1}{5} + \frac{1}{250}x + \frac{3}{25000}x^2$ (b) 0.20020

9 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$r_P = \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ and $r_Q = \begin{pmatrix} 8 \\ -10 \\ -10 \end{pmatrix} + t\begin{pmatrix} -2 \\ 6 \\ 5 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$r_Q - r_P = \begin{pmatrix} 13 \\ -11 \\ -8 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$

Distance^2 = $|r_Q - r_P|^2$, minimised when $(r_Q - r_P) \cdot \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = 0$

$-88 + t(22) = 0$, so t = 4

Then $r_Q - r_P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, with length $\sqrt{2}$

Answer: t = 4, minimum distance = $\sqrt{2}$ m

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(21\mathbf{i} + 27\mathbf{j})$ km relative to a port O, and moves with constant velocity $(\mathbf{i} + 6\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 17 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(21\mathbf{i} + 27\mathbf{j}) + t(\mathbf{i} + 6\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 17 t, so $|(21\mathbf{i} + 27\mathbf{j}) + t(\mathbf{i} + 6\mathbf{j})| = 17\,t$
Squaring and rearranging: $-252t^2 + 366t + 1170 = 0$, which gives t = 3 or a negative value
t = 3 (the other root is negative since B is faster than A)

A is then at $(21\mathbf{i} + 27\mathbf{j}) + 3(\mathbf{i} + 6\mathbf{j}) = (24\mathbf{i} + 45\mathbf{j})$, so $v_B = \frac{1}{3}(24\mathbf{i} + 45\mathbf{j}) = (8\mathbf{i} + 15\mathbf{j})$

Answer: (a) 3 h (b) $(8\mathbf{i} + 15\mathbf{j})$ km h^{-1}