Day 86 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 + 2x - 99 = 0$
passes through P(5, 8). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre $(-1, 0)$. Gradient of radius CP = $\frac{4}{3}$.

Tangent gradient = $-\frac{3}{4}$: $6x + 8y = 94$.

$x = 0$: $y = \frac{47}{4}$. $y = 0$: $x = \frac{47}{3}$.

Area = $\frac{1}{2} \times \frac{47}{3} \times \frac{47}{4}$.

Answer: $\frac{2209}{24}$

2 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

25 recorded lengths (cm) have mean 50 and standard deviation 8.
One value was recorded as 32 but should have been 56.
Find the correct mean and the correct standard deviation.

$\Sigma x = 25 \times 50 = 1250$, so corrected $\Sigma x = 1250 - 32 + 56 = 1274$

Correct mean = $\frac{1274}{25} = 50.96$

$\Sigma x^2 = 25(64 + 2500) = 64100$, corrected $\Sigma x^2 = 64100 - 1024 + 3136 = 66212$

Variance = $\frac{66212}{25} - 50.960^2 = 51.558$, $\sigma = 7.180$

Answer: mean 51.0, standard deviation 7.18

3 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\operatorname{cosec}(2\theta + 25°) = -\sqrt{2}$
for $-90° \leq \theta \leq 270°$.

Let $u = 2\theta + 25°$; then $\operatorname{cosec} u = -\sqrt{2}$
Base solutions: $u = 225°, -45°$ (+ multiples of 360°)
$\theta = (u - 25°) \div 2$, keeping those in range
Answer: θ = −-80°, −-35°, 100°, 145°

4 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An arithmetic sequence has $u_5 = 21$ and $S_{10} = 235$.
(a) Find the first term and common difference.
(b) Find the least n for which $S_n > 855$.

u_5: $a + 4d = 21$. S_{10}: $10(2a + 9d)/2 = 235$, so $10a + 45d = 235$.
Solving: $a = 1$, $d = 5$.
$S_n = \frac{n}{2}(2 + 5n - 5) = \frac{n(5n - 3)}{2}$.
Test: $S_{18} = 783 \leq 855$, $S_{19} = 874 > 855$.
Answer: (a) a = 1, d = 5 (b) n = 19

5 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 8 m s^{-1}. 4 s later, car B leaves O from rest with constant acceleration 1.25 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = $8(t + 4)$; B: distance = $\frac{1}{2} \times 1.25 \times t^2$
Equate: $1.25t^2 - 16t - 64 = 0$, so $(t - 16)(1.25t + 4) = 0$
$t = 16$ (the other root is negative)
OX = $1.25 \times 16^2 / 2 = 160$ m, and speed = $1.25 \times 16 = 20$
Answer: (a) 16 s (b) 160 m (c) 20 m s^{-1}

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = \ln t$, $y = 5t^2 + t$, $t > 0$.
Find the equation of the tangent where $t = 1$, in the form $y = mx + c$.

$dx/dt = \frac{1}{t}$, $dy/dt = 10t + 1$

$dy/dx = t(10t + 1)$ which is 11 at $t = 1$
At $t = 1$: $x = 0$, $y = 6$
Answer: $y = 11x + 6$

7 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 40 N is pulled at constant speed along rough horizontal ground ($\mu = 0.75$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{24}{7}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 40$. Horizontal: $P\cos\theta = \mu R = 0.75(40 - P\sin\theta)$
$P(\cos\theta + 0.75\sin\theta) = 0.75 \times 40$, so $P = 30 \div (\cos\theta + 0.75\sin\theta)$

(a) $\cos\theta = \frac{7}{25}$, $\sin\theta = \frac{24}{25}$: $P = 30 \div 1 = 30$

(b) P is least when $\cos\theta + 0.75\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + 0.75^2)} = 1.25$,
which occurs when $\tan\theta = 0.75$. Then $P = 30 \div 1.25 = 24$ and $\theta = 36.9°$
Answer: (a) 30 N (b) 24 N at $\theta = 36.9°$

8 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{2}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$A + B$ is acute, so $A + B = \pi/4$

Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$

9 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t$, $y = 3t^2$. The line $y = 9x - 24$
meets C at A and B. Find the exact length AB.

Substitute: $3t^2 = 18t - 24$.
$3t^2 - 18t + 24 = 0 \Rightarrow t = 2$ or $t = 4$.
A(4, 12), B(8, 48).
$AB^2 = (-4)^2 + (-36)^2 = 1312$.
Answer: $AB = 4\sqrt{82}$

10 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rare allele is present in 2% of a population. A test is positive for 98% of those with it and for 1% of those without it.
Find P(positive) and P(has it | positive).

$P(\text{positive}) = 0.02 \times 0.98 + 0.98 \times 0.01$
$= 0.0196 + 0.0098 = 0.0294$

$P(\text{has it} \mid \text{positive}) = \frac{0.0196}{0.0294}$

$= 0.6667$
Answer: (a) 0.0294 (b) 0.667