Day 82 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $4\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $4\cos\theta - 3\sin\theta = -1$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{4^2 + 3^2} = 5$ and $\tan\alpha = \frac{3}{4}$, so $\alpha = 36.9°$

So $4\cos\theta - 3\sin\theta \equiv 5\cos(\theta + 36.9°)$

$\cos(\theta + \alpha) = \frac{-1}{5}$

$\theta + \alpha = \pm 101.5°$ (using $\alpha = 36.9°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) $R = 5$, $\alpha = 36.9°$ (b) $\theta = 64.7°, 221.6°$

2 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 12 m s^{-1} from the top of a cliff. The top of the cliff is 8.1 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Take upwards as positive: $-8.1 = 12t - 4.9t^2$, so $4.9t^2 - 12t - 8.1 = 0$

$t = \frac{12 + \sqrt{(12^2 + 19.6 \times 8.1)}}{9.8} = \frac{12 + 17.4}{9.8} = 3$

Height risen above the cliff top $= \frac{12^2}{(2 \times 9.8)} = 7.35$ m

Total distance $= 2 \times 7.35 + 8.1$

Answer: (a) 3 s (b) 22.8 m

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 16 and upper quartile 22. An outlier is a value more than 2 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 8 observations have mean 16. The largest is 37, an outlier, and it is removed. Find the new mean.

IQR $= 22 - 16 = 6$, so $2 \times$ IQR $= 12$

Lower limit $= 16 - 12 = 4$, upper limit $= 22 + 12 = 34$

$\Sigma x = 8 \times 16 = 128$, so without 37: $\Sigma x = 91$

New mean $= \frac{91}{7}$

Answer: (a) 4 to 34 (b) 13.00

4 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(16 + x)^{\frac{1}{2}}$ up to and including the term in x^2.
(b) Use $x = 0.02$ to estimate $\sqrt{16.02}$ to 5 decimal places.

(a) $(16 + x)^{\frac{1}{2}} = 4(1 + \frac{1}{16}x)^{\frac{1}{2}}$.

$= 4 + \frac{1}{8}x - \frac{1}{512}x^2 + \ldots$

(b) $16 + x$ with $x = 0.02$ gives 16.02.

Substitute: $4 + \frac{1}{8}(0.02) + (-\frac{1}{512})(0.02)^2$.

$= 4.00250$ (5 d.p.).

Answer: (a) $4 + \frac{1}{8}x - \frac{1}{512}x^2$ (b) 4.00250

5 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(8, p)$. It is given that $P(X = 2) = \frac{7}{4} \times P(X = 1)$.

(a) Find the value of p.
(b) Find $P(X = 0)$, correct to 3 significant figures.

$\frac{56}{2} p^2(1 - p)^6 = \frac{7}{4} \times 8\, p(1 - p)^7$

Divide by $p(1 - p)^6$: $3.5\, p = \frac{7}{4}(1 - p)$

$(3.5 + \frac{7}{4})p = \frac{7}{4}$

$p = \frac{1}{3}$

$P(X = 0) = (1 - p)^8 = (\frac{2}{3})^8 = 0.0390$

Answer: (a) $p = \frac{1}{3}$ (b) 0.0390

6 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 10 m and mass 30 kg rests on supports at C and D, where AC = 3 m and DB = 2 m. A girl of mass 50 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(3) = 50(x - 8)$, so $x = 8 + \frac{30 \times 3}{50} = 9.8$

Tilting about C (girl before C): $30(2) = 50(3 - x)$, so $x = 3 - \frac{30 \times 2}{50} = 1.8$

Answer: $1.8 \leq x \leq 9.8$

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + 3y = 1$.
Find the equation of the tangent to C at the point (1, 0),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (1, 0): $dy/dx = -\frac{1}{4}$

$y - 0 = (-\frac{1}{4})(x - 1)$, then clear fractions

Answer: $x + 4y - 1 = 0$

8 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$8^x \times 2^y = 1024$

$9^x \div 3^y = 243$

Write each side with prime bases: $1024 = 2^{10}$, $243 = 3^5$.
First: $2^{3x} \times 2^{1y} = 2^{10}$, so $3x + 1y = 10$.
Second: $3^{2x - 1y} = 3^5$, so $2x - 1y = 5$.
Solving the pair of linear equations: $x = 3$, $y = 1$.
Answer: $x = 3$, $y = 1$

9 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the points of inflection of

$y = \frac{6}{x^2 + 27}$.

$dy/dx = -\frac{12x}{(x^2 + 27)^2}$

$d^2y/dx^2 = \frac{12(3x^2 - 27)}{(x^2 + 27)^3}$

Zero when $x^2 = 9$, so $x = \pm 3$, and the sign changes either side

$y = \frac{6}{36} = \frac{1}{6}$

Answer: $(3, \frac{1}{6})$ and $(-3, \frac{1}{6})$

10 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_{10} 5$ is irrational.

Assume $\log_{10} 5 = \frac{m}{n}$, m, n positive integers.

Then $10^{\frac{m}{n}} = 5$, so $10^m = 5^n$.

Prime factors of 10^m and 5^n must be identical,

but 2 divides 10^m but 2 does not divide 5^n.

This is a contradiction, so $\log_{10} 5$ is irrational.

Answer: Proved