Day 80 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = 1$ and $y = 3$, so $p = -1$ and $q = 3$.

$y = \frac{a}{x - 1} + 3$.

P(2, 6): $6 = \frac{a}{1} + 3$, so $a = 3$.

Answer: $a = 3$, $p = -1$, $q = 3$

2 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 7 components, each faulty with probability 0.15, independently. A box is rejected if it has at least 2 faulty components.
(a) Find the probability that a box is rejected.
(b) 13 boxes are inspected. Find the probability that at least 3 are rejected.

Let $X \sim B(7, 0.15)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.2834$
Let Y = number of rejected boxes, $Y \sim B(13, 0.2834)$
$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.759$
Answer: (a) 0.283 (b) 0.759

3 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of $\tan(\tan^{-1}(\frac{1}{5}) + \tan^{-1}(1))$.

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{1}{5} + 1}{1 - \frac{1}{5} \times 1} = \frac{\frac{6}{5}}{\frac{4}{5}}$

Answer: $\frac{3}{2}$

4 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £800 into an account on 1 January.
Interest of 4% is added each 31 December.
Find the first year-end at which the balance exceeds £20,000.

After n years: $800(1.04) + 800(1.04^2) + \ldots + 800(1.04^n)$.

GP: $a = 800 \times 1.04$, $r = 1.04$: $S_n = \frac{800 \times 1.04\,(1.04^n - 1)}{0.04}$.

$S_n > 20000 \Rightarrow 1.04^n > 1.9615 \Rightarrow n > 17.178$.

So n = 18 years.

Answer: after 18 years

5 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 4 kg and 1 kg hang on a light string over a smooth fixed pulley. The 4 kg particle is 3.75 m above the ground; the 1 kg one is on the ground.

Released from rest, find (a) the tension, (b) the greatest height reached by the 1 kg particle. Take $g = 9.8$ m s^{-2}.

$4g - T = 4a$ and $T - 1g = 1a$, so $a = \frac{4-1}{4+1} \times 9.8 = 5.88$

$T = 4(9.8 - 5.88) = 15.68$

The 1 kg particle rises 3.75 m before the 4 kg particle lands, reaching $v^2 = 2 \times 5.88 \times 3.75 = 44.1$

The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{44.1}{19.6} = 2.25$

Greatest height $= 3.75 + 2.25$

Answer: (a) 15.68 N (b) 6 m

6 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 3x^2 + 4x - 8$.

Show that $(x - 1)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(1) = 1 + 3 + 4 - 8 = 0$, so $(x - 1)$ is a factor.

$f(x) = (x - 1)(x^2 + 4x + 8)$.

For $x^2 + 4x + 8 = 0$: $b^2 - 4ac = 16 - 32 = -16 < 0$.

No real roots from the quadratic, so the only real root is $x = 1$.

Answer: Proved; the only real root is $x = 1$

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = \cos x$ and $y = \cos 2x$ meet at $x = 0$ and at $x = \frac{2\pi}{3}$. Find the exact area enclosed between them for $0 \le x \le \frac{2\pi}{3}$.

For $0 < x < \frac{2\pi}{3}$, $\cos x > \cos 2x$

Area $= \int_0^{2\pi/3} (\cos x - \cos 2x)\,dx = [\sin x - \frac{\sin 2x}{2}]$

$= [\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}]$

Answer: $\frac{3}{4}\sqrt{3}$

8 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A spinner is said to show heads with probability 0.2. In 20 trials it does so 1 times.

Test at the 10% level whether the probability is different from 0.2.

H_0: $p = 0.2$, H_1: $p \neq 0.2$ (two-tailed). $X \sim B(20, 0.2)$

1 is below the mean 4, so use the lower tail: $P(X \le 1) = 0.0692$

Each tail is compared with 5%: 0.0692 is greater than 5%

Do not reject H_0: there is insufficient evidence that the probability is different from 0.2

Answer: Do not reject H_0 (p-value 0.0692 against 5%)

9 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \dfrac{1}{16 + 5\sin\theta + 12\cos\theta}$

(a) Write $5\sin\theta + 12\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{5^2 + 12^2} = 13$, $\tan\alpha = \dfrac{12}{5}$, $\alpha = 67.4°$

Denominator $= 16 + 13\sin(\theta + 67.4°)$, which lies between 3 and 29

Greatest $f(\theta) = \dfrac{1}{3} = \dfrac{1}{3}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \dfrac{1}{29} = \dfrac{1}{29}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 67.4° = 270°$, so $\theta = 202.6°$

Answer: (a) $13\sin(\theta + 67.4°)$ (b) greatest $\frac{1}{3}$, least $\frac{1}{29}$, $\theta = 202.6°$

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time $t = 0$ hours, ship A is at the point with position vector $(-40\mathbf{j})$ km relative to a port O, and moves with constant velocity $(5\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Ship B leaves O at $t = 0$ with constant speed 13 km h^{-1} and intercepts A.

Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(-40\mathbf{j}) + t(5\mathbf{i} - 2\mathbf{j})$. B at time t: t v_B (unknown velocity)

At interception B has travelled 13 t, so $|(-40\mathbf{j}) + t(5\mathbf{i} - 2\mathbf{j})| = 13t$

Squaring and rearranging: $-140t^2 + 160t + 1600 = 0$, which gives $t = 4$ or a negative value

$t = 4$ (the other root is negative since B is faster than A)

A is then at $(-40\mathbf{j}) + 4(5\mathbf{i} - 2\mathbf{j}) = (20\mathbf{i} - 48\mathbf{j})$, so $v_B = \frac{1}{4}(20\mathbf{i} - 48\mathbf{j}) = (5\mathbf{i} - 12\mathbf{j})$

Answer: (a) 4 h (b) $(5\mathbf{i} - 12\mathbf{j})$ km h^{-1}