Day 8 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cumulative distribution function (CIE)

The cdf of X is F(x) = 0 for $x < 0$, $F(x) = \frac{x^4}{81}$ for $0 \leq x \leq 3$, and F(x) = 1 for $x > 3$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find P(0.75 < X < 1.5) and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{4}{81}x^3$ for $0 \leq x \leq 3$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 3 $= \frac{12}{5}$

(c) $P = F(1.5) - F(0.75) = 0.05859$

Median m: $F(m) = 1/2$, so $m = 3 \times (1/2)^{1/4}$

$m = 2.523$

Answer: (a) $f(x) = \frac{4}{81}x^3$ (b) $\frac{12}{5}$ (c) 0.0586, median 2.52

2 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $4\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $4\cos\theta - 3\sin\theta = 4$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{4^2 + 3^2} = 5$ and $\tan\alpha = \frac{3}{4}$, so $\alpha = 36.9°$

So $4\cos\theta - 3\sin\theta \equiv 5\cos(\theta + 36.9°)$

$\cos(\theta + \alpha) = \frac{4}{5}$

$\theta + \alpha = \pm 36.9°$ (using $\alpha = 36.9°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 5, α = 36.9° (b) θ = 286.3°, 360.0°

3 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $7k = n^2 + 2$ has no integer solutions (k and n integers).

Consider $n = 7q + r$, with r = 0, 1, ..., 6.

Squaring, $n^2 \equiv r^2 \pmod 7$, so:

$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 2$; $4 \to 2$; $5 \to 4$; $6 \to 1$ ($r \to n^2 \bmod 7$)

So $n^2 + 2 \equiv 2$ or 3 or 4 or 6 (mod 7).

No residue is 0, so $n^2 + 2$ is never a multiple of 7.

Answer: Proved

4 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{2}{x^2 + 5}$ at the point where x = 1

meets the axes at A and B. Find the exact area of triangle OAB,
where O is the origin.

At x = 1: $y = \frac{1}{3}$. Differentiate: gradient $= -\frac{1}{9}$

Tangent: $y - \frac{1}{3} = (-\frac{1}{9})(x - 1)$, so $y = -\frac{1}{9}x + \frac{4}{9}$

y-intercept $\frac{4}{9}$; x-intercept 4

Area $= \frac{1}{2} \times |\frac{4}{9}| \times |4|$

Answer: $\frac{8}{9}$

5 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 15 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): ½ × 5 × 8 + ½(8 − W)(4) − W(6) = 0
20 + 16 − 2W − 6W = 0, so 8W = 36
W = 4.5

v = 0 where the line from (5, 8) to (9, −4.5) crosses the axis: $t = 5 + 4 \times \frac{8}{12.5} = 7.56$

Greatest distance = area of triangle = ½ × 7.56 × 8
Answer: (a) W = 4.5 (b) 30.24 m

6 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the expansion of $(1 + kx)^n$, where n is a positive integer, the coefficient of x is −8 and the coefficient of x^2 is 28. Find n and k.

Coefficient of x: nk = −8. Coefficient of x^2: $\frac{n(n-1)}{2}k^2 = 28$.

Divide the second by $(nk)^2$: $\frac{n-1}{2n} = \frac{28}{64}$.

$2n \times 28 = 64(n-1) \Rightarrow n = \frac{64}{8} = 8$.

Then $k = \frac{-8}{8} = -1$.

Answer: n = 8, k = −1

7 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 6-sided dice, numbered 1 to 6, are rolled. M is the smaller of the two scores.

(a) Show that $P(M = m) = \frac{(13-2m)}{36}$ for m = 1, 2, ..., 6.

(b) Find P(M = 5) and E(M).

(a) M ≥ m means both scores ≥ m: $P(M \geq m) = (\frac{6-m+1}{6})^2$, so $P(M = m) = \frac{(6-m+1)^2 - (6-m)^2}{36} = \frac{(13-2m)}{36}$

(b) $P(M = 5) = \frac{1}{12}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{36}\Sigma\, m(13-2m)$

$= \frac{91}{36}$

Answer: (b) P(M = 5) = $\frac{1}{12}$, E(M) = $\frac{91}{36}$

8 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + 3xy + 3y^2 = 9$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: 2x + by + (bx + 2cy)dy/dx = 0, so $dy/dx = -\frac{2x+3y}{3x+6y}$

Vertical tangent when 3x + 6y = 0, so $x = -\frac{6y}{3}$

Substitute into the equation of C: $x^2 + (3xy) + (3y^2) = 9$
Solve for the remaining coordinate
Answer: (−6, 3) and (6, −3)

9 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{12}{13}$. A force P N acts up a line of greatest slope; the particle stays at rest for $56 \leq P \leq 112$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 224. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 112 and up the plane at P = 56
$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$

Adding: $2W\sin\alpha = 168$, so $W = \frac{168}{2 \times \frac{12}{13}} = 91$

Subtracting: $2F = 56$, so $F = 28 = \mu \times 91 \times \frac{5}{13}$, giving $\mu = 0.8$

P = 224: net force up the plane = 224 − 112 = 112, and mass = $\frac{91}{9.8}$

Answer: (a) 91 N (b) 0.8 (c) 12.1 m s^{-2}

10 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(9 + 5x)^{-\frac{3}{2}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(9 + 5x)^{-\frac{3}{2}} = 9^{-\frac{3}{2}}(1 + \frac{5}{9}x)^{-\frac{3}{2}} = \frac{1}{27}(1 + u)^{-\frac{3}{2}}$, $u = \frac{5}{9}x$.

$(1 + u)^{-\frac{3}{2}} = 1 + (-\frac{3}{2})u + \frac{15}{8}u^2 + (-\frac{35}{16})u^3 + ...$

Substitute u and multiply by $\frac{1}{27}$.

Valid when $|u| < 1$, i.e. $|x| < \frac{9}{5}$.

Answer: $\frac{1}{27} - \frac{5}{162}x + \frac{125}{5832}x^2 - \frac{4375}{314928}x^3 + ...$; $|x| < \frac{9}{5}$