Day 78 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = e^x + 1$ to find the exact value of

$$\int_0^{\ln 7} \frac{e^x}{e^x + 1}\,dx$$

$du = e^x\,dx$; limits $x = 0, \ln 7$ become $u = 2, 8$

Integral $= \int_2^8 \frac{1}{u}\,du = [\ln u]$

Evaluate the limits

Answer: $2\ln 2$

2 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 132.3 m above the ground for 6 s.

Find the speed of projection and the greatest height reached. Take $g = 9.8$ m s^{-2}.

By symmetry, the ball takes 3 s to rise from height 132.3 m to its highest point

Speed at height 132.3 m: $v = 9.8 \times 3 = 29.4$ m s^{-1}

Speed of projection: $u^2 = v^2 + 2gh = 864.36 + 2 \times 9.8 \times 132.3 = 3457.44$

$u = 58.8$. Greatest height $= \frac{u^2}{(2g)} = \frac{3457.44}{19.6} = 176.4$

Answer: 58.8 m s^{-1}, 176.4 m

3 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (2, 2) and (4, 1).

Find the values of a and n.

$2 = a \times 2^n$ and $1 = a \times 4^n$

Divide: $2^n = \frac{1}{2}$ so $n = -1$

$a = 2 \div 2^{-1} = 4$

Answer: a = 4, n = −1

4 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 3$ and $v = \frac{y}{4}$. The regression line of v on u is $v = 6 + 0.5u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 11$.

Substitute $u = x - 3$ and $v = y/4$: $\frac{y}{4} = 6 + 0.5(x - 3)$

$y = 4[6 + 0.5x - 1.5] = 4 \times 4.5 + 4 \times 0.5x$

$y = 18 + 2x$

At $x = 11$: $y = 18 + 2 \times 11 = 40$

Answer: (a) y = 18 + 2x (b) 40

5 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 7 and common ratio $\frac{9}{10}$.

Find the least n for which S_n exceeds 99% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.99\,S_\infty$.

$1 - \left(\frac{9}{10}\right)^n > 0.99$, so $\left(\frac{9}{10}\right)^n < 0.01$.

$n > \frac{\log(0.01)}{\log(0.9)} = 43.709$.

So the least n is 44.

Answer: n = 44

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6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x^2 \ln x$ for $x > 0$,
and determine its nature.

$dy/dx = x^1(2 \ln x + 1) = 0$ gives $\ln x = -\frac{1}{2}$

$y = -\frac{1}{2}e^{-1}$ when $x = \frac{1}{\sqrt{e}}$

Second derivative at this point is positive, so it is a minimum

Answer: $(\frac{1}{\sqrt{e}}, -\frac{1}{2}e^{-1})$, minimum

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=5}^{12}(3 \times 2^r + 2r)$.

Geometric part: first term $3 \times 2^5 = 96$, ratio 2, 8 terms.

Sum = $\frac{96(2^8 - 1)}{1} = 24480$.

Arithmetic part: 2×8 terms, average $(5 + 12)/2$: sum = 136.

Total = 24480 + 136.

Answer: 24616

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - y = -6$
$x^2 + 2y^2 = 27$

From the first equation $x = -6 + y$.
Substitute into the second equation and collect terms:
this gives $3y^2 - 12y + 9 = 0$.
$y = 1$ or $y = 3$, giving $x = -5$ or $x = -3$.

Answer: (x, y) = (−5, 1) and (−3, 3)

9 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) = $\frac{1}{3}$, P(B scores) = $\frac{1}{2}$ each attempt.
Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) = $\frac{1}{3}$; to reach attempt 2 both miss first: $(\frac{2}{3})(\frac{1}{2}) = \frac{1}{3}$

(a) $\frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{4}{9}$

(b) P(A wins) = $\frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3}^2 \times \frac{1}{3} + ...$

Geometric: first term $\frac{1}{3}$, common ratio $\frac{1}{3}$

$S = \frac{a}{1 - r} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

Answer: (a) $\frac{4}{9}$ (b) $\frac{1}{2}$

10 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at ($\mathbf{i} - 4\mathbf{j}$) km relative to a fixed origin and B is at ($-15\mathbf{i} - 17\mathbf{j}$) km. A has velocity ($-2\mathbf{i} - 2\mathbf{j}$) km h^{-1} and B has velocity ($4\mathbf{i} + 6\mathbf{j}$) km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-16\mathbf{i} - 13\mathbf{j}) + t(6\mathbf{i} + 8\mathbf{j})$

$|\mathbf{d}|^2 = (6t - 16)^2 + (8t - 13)^2 = 100t^2 - 400t + 425$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-200}{100} = 2$

$\mathbf{d} = (-16\mathbf{i} - 13\mathbf{j}) + 2(6\mathbf{i} + 8\mathbf{j}) = (-4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 2 h (b) 5 km