Day 77 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 22 cm and its area is 28 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 22$ and $\frac{1}{2}r^2\theta = 28$

$r\theta = 22 - 2r$, so $\frac{1}{2}r(22 - 2r) = 28$

$r^2 - 11r + 28 = 0$, so $(r - 4)(r - 7) = 0$

$r = 4$: $r\theta = 14$, $\theta = \frac{7}{2}$ $r = 7$: $r\theta = 8$, $\theta = \frac{8}{7}$

Answer: $r = 4, \theta = \frac{7}{2}$ or $r = 7, \theta = \frac{8}{7}$

2 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{14}$ is irrational, prove by contradiction that $\sqrt{2} + \sqrt{7}$ is irrational.

Assume $\sqrt{2} + \sqrt{7} = r$, where r is rational.
Square both sides: $2 + 7 + 2\sqrt{14} = r^2$.
So $\sqrt{14} = \frac{r^2 - 9}{2}$.
The right-hand side is rational, but $\sqrt{14}$ is irrational.
This is a contradiction, so $\sqrt{2} + \sqrt{7}$ is irrational.
Answer: Proved

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-3\mathbf{i} - 7\mathbf{j})$ N, $(-2\mathbf{i} + k\mathbf{j})$ N and $(-\mathbf{i} + 2\mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(-\mathbf{i} - \mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(-6)\mathbf{i} + (-5 + k)\mathbf{j}$

Parallel to $(-\mathbf{i} - \mathbf{j})$: $\frac{-5 + k}{-6} = \frac{-1}{-1}$, so $k = -1$

$\mathbf{R} = (-6\mathbf{i} - 6\mathbf{j})$ N, so $|R| = \sqrt{(-6^2 + -6^2)} = 8.485$
$a = |R| \div 4$
Answer: (a) $k = -1$ (b) 2.12 m s^{-2}

4 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$\int_0^{\pi/6} \tan 2x \, dx$

$\tan 2x = \frac{\sin 2x}{\cos 2x}$ and $d/dx(\cos 2x) = -2 \sin 2x$

Integral = $[-\frac{1}{2}\ln|\cos 2x|]$

cos at the upper limit is $\frac{1}{2}$

Answer: $\frac{1}{2}\ln 2$

5 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 90 kg, hangs from a vertical cable. The tension in the cable is 4644 N and the floor exerts a force of 774 N on the passenger.

Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $774 - 90 \times 9.8 = 90a$, so $a = -1.2$

$a = 1.2$ m s^{-2}, so the acceleration is downwards

Lift and passenger: $4644 - (M + 90) \times 9.8 = (M + 90) \times -1.2$

$(M + 90)(9.8 - 1.2) = 4644$, so $M + 90 = \frac{4644}{8.6} = 540$

Answer: (a) 1.2 m s^{-2} downwards (b) M = 450

6 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.65, P(B) = 0.35 and P(A | B) = 0.15.

(a) Find $P(A \cap B)$.

(b) Find $P(A \mid B')$.

$P(A \cap B) = P(A \mid B) \times P(B) = 0.15 \times 0.35 = 0.0525$

$P(A \cup B) = 0.65 + 0.35 - 0.0525 = 0.9475$

$P(B') = 0.65$

$P(A \cap B') = P(A) - P(A \cap B) = 0.5975$

$P(A \mid B') = \frac{0.5975}{0.65}$

Answer: (a) 0.0525 (b) $\frac{239}{260}$

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=5}^{11}(3 \times 3^r + 2r)$.

Geometric part: first term $3 \times 3^5 = 729$, ratio 3, 7 terms.

Sum $= \frac{729(3^7 - 1)}{2} = 796797$.

Arithmetic part: 2×7 terms, average $(5 + 11)/2$: sum = 112.

Total = 796797 + 112.

Answer: 796909

8 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{5}) + \tan^{-1}(\frac{1}{8})$.

$\tan(\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{5})) = \frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{10}} = \frac{7}{9}$

$\tan(\text{sum}) = \frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{72}} = \frac{\frac{65}{72}}{\frac{65}{72}} = 1$

Each angle is below $\pi/4$ and the sum is below $\pi/2$, so the sum is $\pi/4$

Answer: $\frac{\pi}{4}$

9 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2 - 5x + 3$ is translated by $\begin{pmatrix}2\\2\end{pmatrix}$

and then a stretch, scale factor $\frac{1}{3}$, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix}2\\2\end{pmatrix}$: $y = f(x - 2) + 2 = 2x^2 - 13x + 23$.

Then a stretch, scale factor $\frac{1}{3}$, parallel to the x-axis: replace x by 3x: $y = f(3x - 2) + 2$.

Expand and simplify to give the quadratic.

Answer: $y = 18x^2 - 39x + 23$

10 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 0, 1, 2, 3 with $P(X = x) = k(x + 1)$. X_1 and X_2 are independent copies of X.

(a) Find k.

(b) Find $P(X_1 + X_2 = 3)$.

(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(1 + 2 + 3 + 4) = 10k = 1$, so $k = \frac{1}{10}$

(b) Ways to total 3: (0,3), (1,2), (2,1), (3,0)

$P = \frac{1}{10}\times\frac{2}{5} + \frac{1}{5}\times\frac{3}{10} + \frac{3}{10}\times\frac{1}{5} + \frac{2}{5}\times\frac{1}{10} = \frac{1}{5}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{3}{10}$

$E(X) = 2$, so $E(X_1 + X_2) = 2 \times 2 = 4$

Answer: (a) $k = \frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$, 4