Day 76 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_3 x - \log_3 y = 1$
$x + y = 36$

$\log_3(x/y) = 1$, so $x = 3^1 y = 3y$
$3y + y = 36$, so $4y = 36$
Answer: $x = 27$, $y = 9$

2 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 33 m s^{-1}. At the same instant, ball B is released from rest 33 m directly above A.
The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision.
Take g = 9.8 m s^{-2}.

Relative to B, A moves upwards at the constant speed 33 m s^{-1} (both have acceleration g)
So the balls meet when $33\,t = 33$, giving $t = 1$
Height of A: $s = 33 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 28.1$
$v = u - gt = 33 - 9.8 \times 1 = 23.2$
Answer: (a) 28.1 m (b) 23.2 m s^{-1} upwards

3 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 5 cm, subtends an angle of $\frac{\pi}{3}$ at O.
Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 5^2 \times \frac{\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = \frac{25}{4}\sqrt{3}$

Area = $\frac{25}{6}\pi - \frac{25}{4}\sqrt{3}$
Chord AB = $2r\sin(\theta/2) = 5$
Arc AB = $r\theta = \frac{5}{3}\pi$
Perimeter = arc + chord
Answer: Area = $\frac{25}{6}\pi - \frac{25}{4}\sqrt{3}$ cm^2, perimeter = $5 + \frac{5}{3}\pi$ cm

4 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (9, 2).
Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.
Substitute (9, 2): $(9 - r)^2 + (2 - r)^2 = r^2$.
$r^2 - 22r + 85 = 0$.
$(r - 5)(r - 17) = 0$, so $r = 5$ or $r = 17$.
Answer: $(x - 5)^2 + (y - 5)^2 = 25$ or $(x - 17)^2 + (y - 17)^2 = 289$

5 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^3 + xy^2 = 100$.
Find the equation of the tangent to C at the point (4, −3),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x
At (4, −3): $dy/dx = \frac{19}{8}$
$y - -3 = \frac{19}{8}(x - 4)$, then clear fractions
Answer: $19x - 8y - 100 = 0$

6 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 19 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 41,
(b) the least number of rows for at least 611 seats.

Arithmetic: $a = 19$, $d = 3$.
(a) $u_{41} = 19 + 40 \times 3 = 139$.

(b) $S_n = \frac{n}{2}(38 + 3n - 3) = \frac{n(3n + 35)}{2} \geq 611$.

$S_{15} = 600 < 611$, $S_{16} = 664 \geq 611$.

Answer: (a) 139 (b) 16 rows

7 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to show heads with probability 0.5. In 25 trials it does so 20 times.
Test at the 10% level whether the probability is different from 0.5.

H_0: $p = 0.5$, H_1: $p \neq 0.5$ (two-tailed). $X \sim B(25, 0.5)$
20 is above the mean 12.5, so use the upper tail: $P(X \geq 20) = 0.00204$
Each tail is compared with 5%: 0.00204 is less than 5%
Reject H_0: there is sufficient evidence that the probability is different from 0.5

Answer: Reject H_0 (p-value 0.00204 against 5%)

8 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = 1 - (1 - \frac{x}{5})^3$ for $0 \leq x \leq 5$, and $F(x) = 1$ for $x > 5$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find $P(1.25 < X < 2.5)$ and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{3}{5}(1 - \frac{x}{5})^2$ for $0 \leq x \leq 5$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 5 $= \frac{5}{4}$

(c) $P = F(2.5) - F(1.25) = 0.2969$
Median m: $F(m) = 1/2$, so $m = 5(1 - (1/2)^{1/3})$
$m = 1.031$

Answer: (a) $f(x) = \frac{3}{5}(1 - \frac{x}{5})^2$ (b) $\frac{5}{4}$ (c) 0.297, median 1.03

9 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -\frac{1}{2}$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{99} u_r$.

$u_1 = -\frac{1}{2}$, $u_2 = \frac{2}{3}$, $u_3 = 3$, $u_4 = -\frac{1}{2}$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $\frac{19}{6}$.

$99 = 3 \times 33 + 0$: 33 full periods plus the first 0 terms.

Answer: $\frac{209}{2}$

10 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 37 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 74 m horizontally and 26 m vertically from O.

Find the two possible values of tan θ, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 37 \cos\theta \times t$. Vertical: $y = 37 \sin\theta \times t - 4.9t^2$

Eliminating t: $y = x \tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 74^2}{2 \times 37^2} = 19.6$, so $26 = 74 \tan\theta - 19.6(1 + \tan^2\theta)$

$19.6 \tan^2\theta - 74 \tan\theta + 45.6 = 0$

$\tan\theta = \frac{38}{49}$ or 3, so $\theta = 37.8°$ or $71.6°$

Answer: $\tan\theta = \frac{38}{49}$ or 3; $\theta = 37.8°$ or $71.6°$