Day 74 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each battery is faulty with probability 0.03, independently.
Find the least number n of batterys that must be tested so that the probability of finding at least one faulty is greater than 0.995.

P(at least one faulty) = $1 - 0.97^n > 0.995$
$0.97^n < 0.005$
$n \ln 0.97 < \ln 0.005$
Dividing by ln 0.97 (negative) reverses the inequality: $n > 173.9$
$n = 174$
Answer: n = 174

2 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta + 3\cos\theta - 1 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta + 3\cos\theta - 2 = 0$
$(2\cos\theta - 1)(\cos\theta + 2) = 0$
$\cos\theta = \frac{1}{2}$ or $\cos\theta = -2$ (impossible, $|\cos\theta| \leq 1$)
Using the CAST diagram for $0 \leq \theta < 360°$
Answer: θ = 60°, 300°

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 15 m s^{-1} from the top of a cliff. The top of the cliff is 47.5 m above the sea.
Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-47.5 = 15t - 4.9t^2$, so $4.9t^2 - 15t - 47.5 = 0$
$t = \frac{15 + \sqrt{(15^2 + 19.6 \times 47.5)}}{9.8} = \frac{15 + 34}{9.8} = 5$
Height risen above the cliff top = $\frac{15^2}{(2 \times 9.8)} = 11.5$ m
Total distance = $2 \times 11.5 + 47.5$
Answer: (a) 5 s (b) 70.5 m

4 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{18x + 7}{(1 + 2x)(1 + 3x)}$.
(a) Express f(x) in partial fractions.
(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $18x + 7 \equiv A(1 + 3x) + B(1 + 2x)$: A = 4, B = 3.
$f(x) = \frac{4}{1 + 2x} + \frac{3}{1 + 3x}$.
(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use c = 2, then c = 3).
Coefficients of x^j: $4(-2)^j + 3(-3)^j$.
Valid for $|x| < \frac{1}{3}$ (both expansions).
Answer: (a) A = 4, B = 3 (b) $7 - 17x + 43x^2 - 113x^3 + \ldots$; $|x| < \frac{1}{3}$

5 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ladder of length 17 m leans against a vertical wall. Its foot slides away from the wall at 2 m s^{-1}. When the foot is 8 m from the wall, find the rate at which the top is moving down.

$x^2 + y^2 = 289$, so $2x\,dx/dt + 2y\,dy/dt = 0$

When $x = 8$, $y = 15$

$dy/dt = -\frac{x}{y}\,dx/dt = -\frac{8}{15} \times 2$

Answer: $\mathbf{\frac{16}{15}\ m\ s^{-1}}$

6 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.

Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 12 × k = 1, so $k = \frac{1}{6}$

(b) The mean of a triangular distribution is $\frac{0 + 11 + 12}{3} = \frac{23}{3}$

$F(11) = \frac{11}{12} = \frac{11}{12} \geq \frac{1}{2}$, so the median is in the left part

$F(x) = \frac{x^2}{132} = \frac{1}{2}$, so $x = \sqrt{66} = 8.124$

Answer: (a) k = $\mathbf{\frac{1}{6}}$ **(b) E(X) =** $\mathbf{\frac{23}{3}}$ **(c) 8.12**

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−3, 3, 1), (−3, 5, 5), (5, 2, 0).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}$, $BC = \begin{pmatrix} 8 \\ -3 \\ -5 \end{pmatrix}$; $BA \cdot BC = 26$

$|BA| = \sqrt{20}$, $|BC| = \sqrt{98}$

$\cos ABC = \frac{26}{\sqrt{20}\sqrt{98}} = 0.5873$, so ABC = 54.0°

$|BA \times BC| = \left|\begin{pmatrix} -2 \\ -32 \\ 16 \end{pmatrix}\right| = \sqrt{1284}$

Area = $\frac{1}{2}|BA \times BC|$

Answer: ABC = 54.0°, area = $\mathbf{\sqrt{321}}$

8 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.

Line l_2 is perpendicular to l_1 at B and meets the x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{6 - 2}{3 - 9} = -\frac{2}{3}$.

Gradient of $l_2 = \frac{3}{2}$. Equation: $y - 6 = \frac{3}{2}(x - 3)$.

$y = 0$: $x = -1$, so C(−1, 0).

$AB = 2\sqrt{13}$, $BC = 2\sqrt{13}$: area = ½ × AB × BC.

Answer: C(−1, 0); area 26

9 Car on a hill: power and maximum speed (CIE)

A car of mass 1000 kg, with engine power 17.3 kW, climbs a straight road with $\sin\alpha = \frac{1}{25}$ against a constant resistance of 300 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at $\frac{25}{3}$ m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $300 + 1000 \times 9.8 \times \frac{1}{25} = 692$ N

$P = F v$, so $v = \frac{17300}{692} = 25$

At $\frac{25}{3}$ m s^{-1}: $F = \frac{17300}{\frac{25}{3}} = 2076$ N

$F - 692 = 1000\,a$, so $a = \frac{1384}{1000}$

Answer: (a) 25 m s^{-1} (b) 1.38 m s^{-2}

10 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{-2x + 12}{x + 2}$.

(a) Write down the equations of the asymptotes of C.
(b) The line $y = -x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = -2$. As $x \to \infty$, $y \to -2$, so $y = -2$.
(b) $-2x + 12 = (-x + k)(x + 2)$.
$-x^2 + kx + (2k - 12) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 + 8k - 48 = 0$.
Answer: (a) $x = -2$, $y = -2$ (b) $k = -12$ or $k = 4$