Day 73 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 44 cm and its area is 117 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 44$ and $\frac{1}{2}r^2\theta = 117$

$r\theta = 44 - 2r$, so $\frac{1}{2}r(44 - 2r) = 117$

$r^2 - 22r + 117 = 0$, so $(r - 9)(r - 13) = 0$

r = 9: $r\theta = 26$, $\theta = \frac{26}{9}$ r = 13: $r\theta = 18$, $\theta = \frac{18}{13}$

Answer: r = 9, $\theta = \frac{26}{9}$ or r = 13, $\theta = \frac{18}{13}$

2 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 29.4^2 - 2 \times 9.8 \times H$, so H = 44.1

At height $\frac{H}{2}$ = 22.05: $v_y^2 = 29.4^2 - 2 \times 9.8 \times 22.05 = 432.18$

v_x = 39.2 (constant), so $speed^2 = 1536.64 + 432.18 = 1968.82$

Answer: (a) 44.1 m (b) 44.4 m s^{-1}

3 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{x - 3} = x - 5$.

Square both sides: $x - 3 = (x - 5)^2$.

$x^2 - 11x + 28 = 0$, so x = 4 or x = 7.

Check in $\sqrt{x - 3} = x - 5$: x = 7 gives $2 \geq 0$ (valid).

x = 4 gives right-hand side $-1 < 0$, so it is rejected.

Answer: x = 7

4 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 11 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 4 \times 6 + \frac{1}{2}(6 - W)(2) - W(5) = 0$

$12 + 6 - 1W - 5W = 0$, so $6W = 18$

W = 3

v = 0 where the line from (4, 6) to (6, −3) crosses the axis: $t = 4 + 2 \times \frac{6}{9} = \frac{16}{3}$

Greatest distance = area of triangle = $\frac{1}{2} \times \frac{16}{3} \times 6$

Answer: (a) W = 3 (b) 16 m

5 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\tan(\tan^{-1}(\frac{1}{6}) + \tan^{-1}(\frac{1}{2}))$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{6} + \frac{1}{2}}{1 - \frac{1}{6} \times \frac{1}{2}} = \frac{\frac{2}{3}}{\frac{11}{12}}$$

Answer: $\frac{8}{11}$

6 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{5}{x^2 + 8x + 21}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 8x + 21 = (x + 4)^2 + 5$.

$(x + 4)^2 \geq 0$, so the denominator is at least 5, with equality at $x = -4$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{5}{5} = 1$ at $x = -4$.

Answer: greatest value 1 at $x = -4$; range $0 < f(x) \leq 1$

7 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ab^x$. The graph of lg y against x is the straight line shown.

Find a and b to 3 significant figures.

$\lg y = \lg a + x \lg b$. The intercept is $\lg a = 1.2$, so $a = 10^{1.2} = 15.85$

Gradient $= \frac{1.5 - 1.2}{6} = 0.05$

$\lg b = 0.05$ so $b = 10^{0.05} = 1.122$

Answer: a = 15.8, b = 1.12

8 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class Q has 30 students with mean mark 53 and standard deviation 7. Class S has 40 students with mean mark 37 and standard deviation 6.

(a) Find the mean mark of all 70 students.

(b) Find the standard deviation of all 70 students.

Σx for Q $= 30 \times 53 = 1590$, Σx for S $= 40 \times 37 = 1480$

Combined mean $= \frac{3070}{70} = 43.9$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $30(49 + 2809) + 40(36 + 1369) = 141940$

Variance $= \frac{141940}{70} - 43.857^2 = 104.27$, so $\sigma = 10.21$

Answer: (a) 43.9 (b) 10.2

9 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = x^2e^{-3x}$ for $x > 0$,

and determine its nature.

$dy/dx = xe^{-3x}(2 - 3x) = 0$ gives $x = 0$ or $x = \frac{2}{3}$

$y = \frac{4}{9}e^{-2}$ when $x = \frac{2}{3}$

Second derivative at this point is negative, so it is a maximum

Answer: $(\frac{2}{3}, \frac{4}{9}e^{-2})$, maximum

10 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the expansion of $(1 + kx)^n$, where n is a positive integer, the coefficient of x is -12 and the coefficient of x^2 is 54. Find n and k.

Coefficient of x: $nk = -12$. Coefficient of x^2: $\frac{n(n-1)}{2}k^2 = 54$.

Divide the second by $(nk)^2$: $\frac{n - 1}{2n} = \frac{54}{144}$.

$2n \times 54 = 144(n - 1) \Rightarrow n = \frac{144}{36} = 4$.

Then $k = \frac{-12}{4} = -3$.

Answer: n = 4, k = −3