Day 72 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{2x^3 - 6x^2 - 12x + 37}{(x-2)(x-3)}$ in the form $ax + b + \frac{C}{(x-2)} + \frac{D}{(x-3)}$.

Divide: the numerator has degree 3, the denominator degree 2.

$(2x^3 - 6x^2 - 12x + 37) \div (x^2 - 5x + 6)$ gives quotient $2x + 4$.

Remainder: $-4x + 13$.

Put $x = 2$: remainder $= C(-1)$ so $C = -5$. Put $x = 3$: $D = 1$.

Answer: $2x + 4 - \frac{5}{(x-2)} + \frac{1}{(x-3)}$

2 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 39.2 m s^{-1}.

Find the length of time for which the ball is more than 73.5 m above the ground, and the greatest height it reaches. Take $g = 9.8$ m s^{-2}.

At height 73.5 m: $v^2 = 39.2^2 - 2 \times 9.8 \times 73.5 = 96.04$, so $v = 9.8$

Time from this height to the top: $t = \frac{9.8}{9.8} = 1$ s

By symmetry the time above this height is $2 \times 1 = 2$ s

Greatest height $= \frac{39.2^2}{(2 \times 9.8)} = 78.4$

Answer: 2 s, 78.4 m

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 29 and upper quartile 46. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 10 observations have mean 36. The largest is 78, an outlier, and it is removed. Find the new mean.

IQR = 46 − 29 = 17, so 1.5 × IQR = 25.5

Lower limit = 29 − 25.5 = 3.5, upper limit = 46 + 25.5 = 71.5

$\Sigma x = 10 \times 36 = 360$, so without 78: $\Sigma x = 282$

New mean $= \frac{282}{9}$

Answer: (a) 3.5 to 71.5 (b) 31.33

4 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 7 s, P is back at its starting point.

Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 2 \times 12 + \frac{1}{2}(12 - W)(2) - W(3) = 0$

$12 + 12 - 1W - 3W = 0$, so $4W = 24$

$W = 6$

$v = 0$ where the line from (2, 12) to (4, −6) crosses the axis: $t = 2 + 2 \times \frac{12}{18} = \frac{10}{3}$

Greatest distance = area of triangle $= \frac{1}{2} \times \frac{10}{3} \times 12$

Answer: (a) W = 6 (b) 20 m

5 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 200 students in Year 10, 100 in Year 11 and 60 in the sixth form.
A stratified sample of 36 students is taken.
(a) Find the number sampled from each group.
(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $36 \times \frac{200}{360}$, $36 \times \frac{100}{360}$, $36 \times \frac{60}{360}$ = 20, 10, 6

P(same) = 20 × 19 + 10 × 9 + 6 × 5/36 × 35

$= \frac{500}{1260} = 0.397$

Answer: (a) 20, 10, 6 (b) 0.397

6 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{2}u_n + 5$, $u_1 = -1$.
(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{1}{2}L + 5$, so $L(1 - \frac{1}{2}) = 5$.

L = 10.

(b) $u_n - L = (u_1 - L)(\frac{1}{2})^{n-1} = (-11)(\frac{1}{2})^{n-1}$.

Need $11 \times (\frac{1}{2})^{n-1} < 0.001$, so $n - 1 > 13.425$.

The least n is 15.

Answer: (a) L = 10 (b) n = 15

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−2, −2, 1), (2, 3, −1), (2, −3, −2).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$, $BC = \begin{pmatrix} 0 \\ -6 \\ -1 \end{pmatrix}$; $BA \cdot BC = 28$

$|BA| = \sqrt{45}$, $|BC| = \sqrt{37}$

$\cos ABC = \frac{28}{\sqrt{45}\sqrt{37}} = 0.6862$, so ABC = 46.7°

$|BA \times BC| = \left|\begin{pmatrix} 17 \\ -4 \\ 24 \end{pmatrix}\right| = \sqrt{881}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 46.7°, area $= \frac{1}{2}\sqrt{881}$

8 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 4$ to find the exact value of

$\int_0^2 x(x^2 + 4)^4 \, dx$

du = 2x dx; limits x = 0, 2 become u = 4, 8

Integral $= \frac{1}{2}\int_4^8 u^4 \, du = [\frac{u^5}{10}]$

Evaluate the limits

Answer: $\frac{15872}{5}$

9 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 6 and common ratio 5.
Find the least n for which $S_n > 7666$.

$S_n = \frac{6(5^n - 1)}{4} > 7666.$

$5^n > 5111.67.$

$n > \frac{\log(5112)}{\log(5)} = 5.306.$

So the least n is 6.

Answer: n = 6

10 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{5}$ and $\tan B = \frac{2}{3}$.

Find tan(A + B) and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{2}{15}} = \frac{\frac{13}{15}}{\frac{13}{15}} = 1$$

A + B is acute, so $A + B = \pi/4$

Answer: tan(A + B) = 1, A + B = $\frac{\pi}{4}$