Day 69 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1200 kg tows a trailer of mass 400 kg along a horizontal road, with driving force 1760 N. Resistances are 800 N on the car and 320 N on the trailer.

(a) Find the tension in the tow bar. At 4.8 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1760 - 800 - 320}{1600} = 0.4$

Trailer: $T - 320 = 400 \times 0.4$, so $T = 480$

After the break the trailer decelerates at $\frac{320}{400} = 0.8$, and stops after $t = \frac{4.8}{0.8} = 6$ s

Trailer travels $\frac{4.8}{2} \times 6 = 14.4$ m

Car: $a = \frac{1760 - 800}{1200} = 0.8$, so $s = 4.8 \times 6 + ½ \times 0.8 \times 6^2 = 43.2$ m

Distance apart = $43.2 - 14.4$

Answer: (a) 480 N (b) 28.8 m

2 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = \frac{3}{2}$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{99} u_r$.

$u_1 = \frac{3}{2}$, $u_2 = -2$, $u_3 = \frac{1}{3}$, $u_4 = \frac{3}{2}$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $-\frac{1}{6}$.

$99 = 3 \times 33 + 0$: 33 full periods plus the first 0 terms.

Answer: $-\frac{11}{2}$

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 11 m s^{-1} from the top of a cliff. The top of the cliff is 67.5 m above the sea.

Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-67.5 = 11t - 4.9t^2$, so $4.9t^2 - 11t - 67.5 = 0$

$t = \frac{11 + \sqrt{(11^2 + 19.6 \times 67.5)}}{9.8} = \frac{11 + 38}{9.8} = 5$

Height risen above the cliff top = $\frac{11^2}{(2 \times 9.8)} = 6.17$ m

Total distance = $2 \times 6.17 + 67.5$

Answer: (a) 5 s (b) 79.8 m

4 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{3}{5}$, independently.

(a) Find the probability that her third score comes on shot number 9.
(b) Find the probability that her third score comes on or before shot number 9.

(a) third score on shot 9: exactly 2 scores in the first 8 shots, then a score

$C(8, 2)(\frac{3}{5})^2(\frac{2}{5})^6 \times \frac{3}{5}$

$= 0.0248$

(b) On or before shot 9 means at least 3 scores in 9 shots: $X \sim B(9, \frac{3}{5})$

$P(X \geq 3) = 1 - P(X \leq 2) = 0.975$

Answer: (a) 0.0248 (b) 0.975

5 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{x + 7} = x + 5$.

Square both sides: $x + 7 = (x + 5)^2$.

$x^2 + 9x + 18 = 0$, so $x = -6$ or $x = -3$.

Check in $\sqrt{x + 7} = x + 5$: $x = -3$ gives $2 \geq 0$ (valid).

$x = -6$ gives right-hand side $-1 < 0$, so it is rejected.

Answer: $x = -3$

6 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 5$ and $v = \frac{y}{10}$. The regression line of v on u is $v = 14.5 + 1.8u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 9$.

Substitute $u = x - 5$ and $v = y/10$: $\frac{y}{10} = 14.5 + 1.8(x - 5)$

$y = 10[14.5 + 1.8x - 9] = 10 \times 5.5 + 10 \times 1.8x$

$y = 55 + 18x$

At $x = 9$: $y = 55 + 18 \times 9 = 217$

Answer: (a) $y = 55 + 18x$ (b) 217

7 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = 1 + \ln x$ to find the exact value of

$$\int_1^{e^4} \frac{1}{x(1 + \ln x)}\, dx$$

$du = \frac{1}{x}\, dx$; limits $x = 1, e^4$ become $u = 1, 5$

Integral $= \int_1^5 \frac{1}{u}\, du = [\ln u]$

Evaluate the limits

Answer: ln 5

8 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2$, $y = 2t(3 - t)$,
for $0 \leq t \leq 3$. The curve meets the x-axis at O and at A.
Find the area enclosed between the curve and the x-axis.

$dx/dt = 6t$; x from 0 to 27 as t goes from 0 to 3

Area $= \int_0^3 y \frac{dx}{dt}\, dt = \int_0^3 12t^2(3 - t)\, dt$

$= 12[\frac{3t^3}{3} - \frac{t^4}{4}]$

Answer: 81

9 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 6 and common ratio 2.
Find the least n for which $S_n > 81260$.

$S_n = 6(2^n - 1) > 81260$.

$2^n > 13544.33$.

$n > \frac{\log(1.354e+04)}{\log(2)} = 13.725$.

So the least n is 14.

Answer: n = 14

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix}0\\7\\3\end{pmatrix} + t\begin{pmatrix}1\\-3\\1\end{pmatrix}$.

The point P has coordinates (−1, 0, 5).
Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix}-1\\-7\\2\end{pmatrix} - t\begin{pmatrix}1\\-3\\1\end{pmatrix}$

FP · d = 0: 22 − t(11) = 0, so t = 2

F = (2, 1, 5)

$\mathrm{PF} = \begin{pmatrix}-3\\-1\\0\end{pmatrix}$, so $\mathrm{PF} = \sqrt{10}$

Answer: F = (2, 1, 5), PF = $\sqrt{10}$