Day 66 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\cos 3\theta - \cos 1\theta}{\theta^2}$ approaches as $\theta \to 0$.

$\cos 3\theta \approx 1 - \frac{9}{2}\theta^2$ and $\cos 1\theta \approx 1 - \frac{1}{2}\theta^2$

Difference $\approx -4\theta^2$, so the expression ≈ -4

Answer: -4

2 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find the horizontal distance between the two points at which the ball is 39.2 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $39.2 = 29.4\,t - 4.9t^2$, i.e. $4.9t^2 - 29.4t + 39.2 = 0$

$t = \frac{29.4 \pm \sqrt{(29.4^2 - 19.6 \times 39.2)}}{9.8} = \frac{29.4 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)

Horizontal distance = 39.2 × 2

Answer: 78.4 m

3 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n(n + 7)(n + 14)(n + 21) + 2401$ is a perfect square for every integer n.

Pair the factors: $n(n + 21) = n^2 + 21n$ and $(n + 7)(n + 14) = n^2 + 21n + 98$.

Let $u = n^2 + 21n$. Expression $= u(u + 98) + 2401$

$= u^2 + 98u + 2401 = (u + 49)^2$

$= (n^2 + 21n + 49)^2$, a perfect square.

Answer: Proved

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 2x + 2$ has a root α.

(a) Show that the equation can be written as $x = \sqrt[3]{(2x + 2)}$.

(b) Use $x_{n+1} = \sqrt[3]{(2x_n + 2)}$ with $x_0 = 1$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(2x + 2)}$

$x_1 = 1.5874$, $x_2 = 1.7297$, $x_3 = 1.7608$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 1.769$

Answer: $x_1 = 1.5874$, $x_2 = 1.7297$, $x_3 = 1.7608$; $\alpha = 1.769$

5 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(14, \frac{1}{4})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{14 - r}{r + 1} \times \frac{1}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(14, r + 1)}{C(14, r)} \times \frac{p}{q} = \frac{14 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{4}$ and $q = \frac{3}{4}$, $p \div q = \frac{1}{3}$

The ratio exceeds 1 when $(14 - r)\frac{1}{4} > (r + 1)\frac{3}{4}$, that is $r < 2.75$

So probabilities rise up to r = 3 and then fall; the mode is 3

P(X = 3) = 0.240

Answer: 3 with probability 0.240

6 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$, solve
$\sin 3\theta = 2\sin\theta$ for $0 < \theta < 2\pi$, in terms of π.

Substitute: $\sin\theta(3 - 4\sin^2\theta - 2) = 0$

$\sin\theta = 0$ or $\sin^2\theta = \frac{1}{4}$

$\sin\theta = 0$ gives π

$\sin^2\theta = \frac{1}{4}$ gives $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Answer: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

7 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 6-sided dice, numbered 1 to 6, are rolled. M is the smaller of the two scores.

(a) Show that $P(M = m) = \frac{(13 - 2m)}{36}$ for $m = 1, 2, ..., 6$.

(b) Find $P(M = 2)$ and $E(M)$.

(a) $M \geq m$ means both scores $\geq m$: $P(M \geq m) = (\frac{6 - m + 1}{6})^2$, so $P(M = m) = \frac{(6 - m + 1)^2 - (6 - m)^2}{36} = \frac{(13 - 2m)}{36}$

(b) $P(M = 2) = \frac{1}{4}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{36}\Sigma\, m(13 - 2m)$

$= \frac{91}{36}$

Answer: (b) $P(M = 2) = \frac{1}{4}$, $E(M) = \frac{91}{36}$

8 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The roots of $x^2 + kx + 24 = 0$ differ by 5.
Find the possible values of k.

Let the roots be α and $\alpha + 5$.

Product: $\alpha(\alpha + 5) = 24$, so $\alpha^2 + 5\alpha - 24 = 0$.

$(\alpha - 3)(\alpha + 8) = 0$, so $\alpha = 3$ or $\alpha = -8$.

Sum: $2\alpha + 5 = -k$, so $k = -(2\alpha + 5) = -11$ or 11.

Answer: $k = \pm 11$

9 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where
$v = 25\sqrt{t} - 5t$.
Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{25}{2\sqrt{t}} - 5 = 0$ when $\sqrt{t} = \frac{5}{2}$, so $t = \frac{25}{4}$

$v = 25 \times \frac{5}{2} - 5 \times \frac{25}{4} = \frac{125}{4}$

$v = 0$ when $\sqrt{t}(25 - 5\sqrt{t}) = 0$: $t = 25$ ($t = 0$ is the start)

$s = \int v\, dt = \frac{2}{3} \times 25\, t^{\frac{3}{2}} - \frac{5}{2}t^2$, so at $t = 25$: $s = \frac{6250}{3} - \frac{3125}{2}$

Answer: (a) $\frac{125}{4}$ m s^{-1} (b) $\frac{3125}{6}$ m

10 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 3 and common ratio 4.
Find the least n for which $S_n > 73515$.

$S_n = \frac{3(4^n - 1)}{3} > 73515.$

$4^n > 73516.$

$n > \frac{\log(7.352e+04)}{\log(4)} = 8.083.$

So the least n is 9.

Answer: n = 9