Day 64 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (2 m, 40 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{5}{12}$, where α is the angle between string and rod. A 40 N particle hangs at P, AP = 0.5 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Moments about A: $T\sin\alpha \times 2 = 40 \times 1 + 40 \times 0.5$

$T \times \frac{5}{13} \times 2 = 60$, so $T = 78$

Horizontal: $H = T\cos\alpha = 72$. Vertical: $V + T\sin\alpha = 40 + 40$, so $V = 50$

Force $= \sqrt{(72^2 + 50^2)} = 87.7$, angle $= \tan^{-1}(\frac{50}{72}) = 34.8°$ above the horizontal

Answer: (a) 78 N (b) 87.7 N at 34.8° above the horizontal (away from the wall)

2 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are modelled by $N(61, \sigma^2)$. The probability of a value above 73 is 0.10.

(a) Find σ.

(b) Find the probability of a value below 70.

$P(X > 73) = 0.10$ so $Z = \frac{73 - 61}{\sigma} = 1.2816$

$\sigma = \frac{12}{1.2816} = 9.364$

$z = \frac{70 - 61}{9.364} = 0.961$

$P(X < 70) = P(Z < 0.961) = 0.832$

Answer: (a) 9.36 (b) 0.832

3 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 5}{x - 6}$ for $x \neq 6$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 6) = kx - 5$, so $x(y - k) = 6y - 5$.

$f^{-1}(x) = \frac{6x - 5}{x - k}$.

$f = f^{-1}$ requires $k = 6$.

$f(x) = x$: $6x - 5 = x(x - 6)$, so $x^2 - 12x + 5 = 0$.

$x = 6 \pm \sqrt{31} = 6 \pm \sqrt{31}$.

Answer: $k = 6$; $x = 6 \pm \sqrt{31}$

4 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + x - 7$.

(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.

(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

$f(1) = -5$ and $f(2) = 3$: a change of sign, f continuous

$f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{3.000}{13.00} = 1.7692$

$x_2 = 1.7692 - \frac{0.3072}{10.39} = 1.7397$

Answer: $x_1 = 1.7692$, $x_2 = 1.7397$

5 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{7}$ is irrational.

Assume $\sqrt{7} = \frac{a}{b}$ with a, b integers and no common factor.

Then $7b^2 = a^2$, so a^2 is a multiple of 7.
As 7 is prime, a is a multiple of 7: let $a = 7k$.
Then $7b^2 = 49k^2$, so $b^2 = 7k^2$, and b is a multiple of 7.
a and b have common factor 7: a contradiction. Hence $\sqrt{7}$ is irrational.
Answer: Proved

6 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-0.5t^2 - t)\mathbf{i} + (-2t^2 + 2t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-t - 1)\mathbf{i} + (-4t + 2)\mathbf{j}$
Parallel to $(\mathbf{i} + \mathbf{j})$: $(-t - 1) \times 1 = (-4t + 2) \times 1$
$t(3) = 3$, so $t = 1$
At $t = 1$: $\mathbf{v} = (-2\mathbf{i} - 2\mathbf{j})$, speed $= \sqrt{(-2^2 + -2^2)}$
Answer: (a) $t = 1$ (b) 2.83 m s^{-1}

7 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (1, 12) and (2, 48).
Find the values of a and n.

$12 = a \times 1^n$ and $48 = a \times 2^n$
Divide: $2^n = 4$ so $n = 2$
$a = 12 \div 1^2 = 12$
Answer: $a = 12$, $n = 2$

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$$\int_0^{\pi/9} \sec^2 3x + \sin 3x \cos 3x \, dx$$

Write sin cos as half of sin of the double angle
Integrate term by term
Substitute the limits
Answer: $\frac{1}{8} + \frac{1}{3}\sqrt{3}$

9 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{-2}{(x - 1)} > \frac{4}{(x - 4)}$.

Multiply by the positive quantity $(x - 1)^2(x - 4)^2$:
$(-2)(x - 1)(x - 4)^2 > 4(x - 4)(x - 1)^2$.
Move to one side and factorise: $(x - 1)(x - 4)(-6x + 12) > 0$.
Critical values: 1, 4, 2. Test the four regions.
Answer: $x < 1$ or $2 < x < 4$

10 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) = $\frac{2}{5}$, P(B scores) = $\frac{1}{4}$ each attempt.

Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) = $\frac{2}{5}$; to reach attempt 2 both miss first: $(\frac{3}{5})(\frac{3}{4}) = \frac{9}{20}$

(a) $\frac{2}{5} + \frac{9}{20} \times \frac{2}{5} = \frac{29}{50}$

(b) P(A wins) = $\frac{2}{5} + \frac{9}{20} \times \frac{2}{5} + \frac{9}{20}^2 \times \frac{2}{5} + ...$

Geometric: first term $\frac{2}{5}$, common ratio $\frac{9}{20}$

$S = \frac{a}{1 - r} = \frac{\frac{2}{5}}{\frac{11}{20}} = \frac{8}{11}$

Answer: (a) $\frac{29}{50}$ (b) $\frac{8}{11}$