Day 63 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 60 m from O, and reaches a maximum height of 15 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 15$, so $u_y = 17.15$

Time of flight = $\frac{2u_y}{9.8}$ = 3.499 s, so $u_x = \frac{60}{3.499}$ = 17.15

$\tan\theta = \frac{u_y}{u_x}$ = 1.000, so θ = 45.0°

Speed = $\sqrt{(u_x^2 + u_y^2)}$ = 24.2

Answer: θ = 45.0°, speed 24.2 m s^{-1}

2 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(1, 8) lies on the circle: $(1 - r)^2 + (8 - r)^2 = r^2$.

$r^2 - 18r + 65 = 0$, so r = 5 or r = 13.

The diagram shows P on the smaller circle, so r = 5.

$(x - 5)^2 + (y - 5)^2 = 25$, then expand.

Answer: $x^2 + y^2 - 10x - 10y + 25 = 0$

3 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_1^2 x \ln x \, dx$

Let u = ln x and dv/dx = x, so du/dx = $\frac{1}{x}$, v = $\frac{x^2}{2}$

Integral = $[\frac{x^2}{2} \ln x] - \int \frac{x^1}{2} dx$

= $[\frac{x^2}{2} \ln x - \frac{x^2}{4}]$

Answer: $-\frac{3}{4} + 2\ln 2$

4 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^3 + 2n$ is divisible by 3
for every positive integer n.

$n^3 + 2n = (n - 1)n(n + 1) + 3n$.

One of three consecutive integers is a multiple of 3,
so $(n - 1)n(n + 1)$ is divisible by 3.

3 is a multiple of 3, so 3n is too. Hence the sum is divisible by 3.

Answer: Proved

5 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $3\text{cosec}^2\theta + 4\cot\theta - 7 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $3\cot^2\theta + 4\cot\theta - 4 = 0$

$(3\cot\theta - 2)(\cot\theta + 2) = 0$

$\cot\theta = \frac{2}{3}$ or $\cot\theta = -2$

$\tan\theta = \frac{3}{2}$ or $-\frac{1}{2}$, so θ = 56.3° and 153.4° (+180°)

Answer: θ = 56.3°, 153.4°, 236.3°, 333.4°

6 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 8 m. After each bounce it rises to $\frac{3}{5}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.01 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 8 + \frac{2 \times 8 \times \frac{3}{5}}{1 - \frac{3}{5}} = 32.$

(b) Rebound height after n bounces $= 8(\frac{3}{5})^n < 0.01.$

$n > \frac{\log(0.00125)}{\log(0.6)} = 13.09$, so n = 14.

Answer: (a) 32 m (b) 14 bounces

7 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{1}{4}$, independently.

(a) Find the probability that her third score comes on shot number 7.
(b) Find the probability that her third score comes on or before shot number 7.

(a) third score on shot 7: exactly 2 scores in the first 6 shots, then a score

$C(6, 2)(\frac{1}{4})^2(\frac{3}{4})^4 \times \frac{1}{4}$

$= 0.0742$

(b) On or before shot 7 means at least 3 scores in 7 shots: $X \sim B(7, \frac{1}{4})$

$P(X \geq 3) = 1 - P(X \leq 2) = 0.244$

Answer: (a) 0.0742 (b) 0.244

8 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 8 components, each faulty with probability 0.1, independently. A box is rejected if it has at least 2 faulty components.

(a) Find the probability that a box is rejected.
(b) 12 boxes are inspected. Find the probability that at least 3 are rejected.

Let $X \sim B(8, 0.1)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.1869$
Let Y = number of rejected boxes, $Y \sim B(12, 0.1869)$
$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.395$

Answer: (a) 0.187 (b) 0.395

9 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-2\mathbf{i} - 3\mathbf{j})$ N, $(-3\mathbf{i} + k\mathbf{j})$ N and $(-8\mathbf{j})$ N act on a particle of mass 3 kg. The resultant is parallel to $(-5\mathbf{i} - 11\mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant $= (-5)\mathbf{i} + (-11 + k)\mathbf{j}$

Parallel to $(-5\mathbf{i} - 11\mathbf{j})$: $\frac{-11 + k}{-5} = \frac{-11}{-5}$, so k = 0

$R = (-5\mathbf{i} - 11\mathbf{j})$ N, so $|R| = \sqrt{(-5^2 + -11^2)} = 12.08$
$a = |R| \div 3$

Answer: (a) k = 0 (b) 4.03 m s^{-2}

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$.

The point P has coordinates $(-5, 2, 2)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathbf{FP} = \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} - t\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$\mathbf{FP} \cdot \mathbf{d} = 0$: $19 - t(19) = 0$, so $t = 1$

$F = (-2, 0, -1)$

$\mathbf{PF} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$, so $PF = \sqrt{22}$

Answer: F = $(-2, 0, -1)$, PF = $\sqrt{22}$