Day 62 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{94 - 42\sqrt{5}}$ in the form
$a - b\sqrt{5}$, where a and b are positive integers.

Suppose $\sqrt{94 - 42\sqrt{5}} = a - b\sqrt{5}$. Square both sides.
$94 - 42\sqrt{5} = a^2 + 5b^2 - 2ab\sqrt{5}$.
Equate: $a^2 + 5b^2 = 94$ and $2ab = 42$, so $ab = 21$.
Try $a = 7$, $b = 3$: $49 + 45 = 94$ ☐.
($a > b\sqrt{5}$, so the root is positive.)
Answer: $7 - 3\sqrt{5}$

2 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each component is faulty with probability 0.02, independently.
Find the least number n of components that must be tested so that the probability of finding at least one faulty is greater than 0.999.

P(at least one faulty) $= 1 - 0.98^n > 0.999$
$0.98^n < 0.001$
$n \ln 0.98 < \ln 0.001$
Dividing by ln 0.98 (negative) reverses the inequality: $n > 341.9$
$n = 342$
Answer: $n = 342$

3 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 4 and common ratio $\frac{2}{3}$.
Find the least n for which S_n exceeds 95% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.95\, S_\infty$.
$1 - \left(\frac{2}{3}\right)^n > 0.95$, so $\left(\frac{2}{3}\right)^n < 0.05$.
$n > \frac{\log(0.05)}{\log(0.666667)} = 7.388$.
So the least n is 8.
Answer: $n = 8$

4 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve $y = \frac{8}{x}$ and the line $y = 6 - x$.
Find the exact area of the shaded region enclosed by the curve and the line.

Intersect: $\frac{8}{x} = 6 - x$ gives $x^2 - 6x + 8 = 0$, so $x = 2$ and $x = 4$
Area $= \int_2^4 \left(6 - x - \frac{8}{x}\right) dx = \left[6x - \frac{x^2}{2} - 8 \ln x\right]$
$= (24 - 8 - 8 \ln 4) - (12 - 2 - 8 \ln 2)$
Answer: $6 - 8\ln 2$

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5 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{6}{1-5} = -\frac{3}{2}$.

Gradient of $l_2 = \frac{2}{3}$. Equation: $y - 6 = \frac{2}{3}(x - 1)$.

$y = 0$: $x = -8$, so C$(-8, 0)$.

AB $= 2\sqrt{13}$, BC $= 3\sqrt{13}$: area = ½ × AB × BC.

Answer: C$(-8, 0)$; area 39

6 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 100 m wide and flows at 30 m s^{-1}. A boatman can row at 34 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: $34 \sin \varphi = 30$, so $\varphi = 61.9°$
Angle to the bank = $90° - \varphi = 28.1°$
Speed across = $\sqrt{(34^2 - 30^2)} = 16$ m s^{-1}
Time = $\frac{100}{16} = 6.25$ s

Answer: (a) 28.1° (b) 6.25 s

7 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 18\, e^{-x} = 9$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 9e^x + 18 = 0$
$(e^x - 6)(e^x - 3) = 0$
$e^x = 3$ or $e^x = 6$

Answer: $x = \ln 3$ and $x = \ln 6$

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$$\int_0^{\pi/12} \tan^2 3x \, dx$$

Use $\tan^2 A \equiv \sec^2 A - 1$
Integrate term by term
Substitute the limits

Answer: $\frac{1}{3} - \frac{1}{12}\pi$

9 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 15 and variance 10.5.
(a) Find n and p.
(b) Find P(X = 17), correct to 3 significant figures.

$np = 15$ and $np(1 - p) = 10.5$

Divide: $1 - p = 10.5 \div 15 = \frac{7}{10}$, so $p = \frac{3}{10}$

$n = 15 \div \frac{3}{10} = 50$

$P(X = 17) = C(50, 17)\, (\frac{3}{10})^{17}\, (\frac{7}{10})^{33} = 0.0983$

Answer: (a) $n = 50$, $p = \frac{3}{10}$ (b) 0.0983

10 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 51 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{8}{15}$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{15}{8}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 51$. Horizontal: $P\cos\theta = \mu R = \frac{8}{15}(51 - P\sin\theta)$

$P(\cos\theta + \frac{8}{15}\sin\theta) = \frac{8}{15} \times 51$, so $P = 27.2 \div (\cos\theta + \frac{8}{15}\sin\theta)$

(a) $\cos\theta = \frac{8}{17}$, $\sin\theta = \frac{15}{17}$: $P = 27.2 \div \frac{16}{17} = 28.9$

(b) P is least when $\cos\theta + \frac{8}{15}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{8}{15}^2)} = \frac{17}{15}$,

which occurs when $\tan\theta = \frac{8}{15}$. Then $P = 27.2 \div \frac{17}{15} = 24$ and $\theta = 28.1°$

Answer: (a) 28.9 N (b) 24 N at θ = 28.1°