Day 60 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 6 and its common difference d is non-zero. The 1st, 3rd and 11th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: 6, 6 + 2d, 6 + 10d.

Geometric: $(6 + 2d)^2 = (6)(6 + 10d)$.

Expand: $4d^2 + 24d + 36 = 60d + 36$, so $d(4d - 36) = 0$.

$d \neq 0$, so $d = 9$. Ratio = $\frac{24}{6} = 4$.

Answer: d = 9, ratio = 4

2 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations

$\log_3 x - \log_3 y = 3$

$x + y = 56$

$\log_3(x/y) = 3$, so $x = 3^3 y = 27y$

$27y + y = 56$, so $28y = 56$

Answer: x = 54, y = 2

3 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(5, 3) and B(4, −4). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - 5)^2 + 9 = (c - 4)^2 + 16$.

Expand and solve: $-2c = -2$, so $c = 1$.

Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$

$= \frac{1}{2}|4 \times (-4) - 3 \times 3| = \frac{25}{2}$.

Answer: C(1, 0); area $\frac{25}{2}$

4 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the top of a vertical cliff, 49 m high, at 24.5 m s^{-1} at angle α above the horizontal ($\tan \alpha = \frac{3}{4}$), and lands on horizontal ground.

Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-49 = 14.7t - 4.9t^2$, so $4.9t^2 - 14.7t - 49 = 0$

t = 5 s (the positive root)

Horizontal distance = 19.6 × 5 = 98 m

On landing: $v_x = 19.6$, $v_y = 14.7 - 9.8 \times 5 = -34.3$

speed = $\sqrt{(19.6^2 + 34.3^2)} = 39.5$, tan angle = $\frac{34.3}{19.6}$, angle = 60.3° below the horizontal

Answer: (a) 5 s, 98 m (b) 39.5 m s^{-1} at 60.3° below the horizontal

5 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(9 + x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use x = 0.01 to estimate $\frac{1}{\sqrt{9.01}}$ to 5 decimal places.

(a) $(9 + x)^{-\frac{1}{2}} = \frac{1}{3}(1 + \frac{1}{9}x)^{-\frac{1}{2}}$.

$= \frac{1}{3} - \frac{1}{54}x + \frac{1}{648}x^2 + ...$

(b) 9 + x with x = 0.01 gives 9.01.

Substitute: $\frac{1}{3} + (-\frac{1}{54})(0.01) + \frac{1}{648}(0.01)^2$.

= 0.33315 (5 d.p.).

Answer: (a) $\frac{1}{3} - \frac{1}{54}x + \frac{1}{648}x^2$ (b) 0.33315

6 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (5 kg) and B (3 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.2$, and B is 1.6 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.2 \times 5 \times 9.8 = 5a$. B: $3 \times 9.8 - T = 3a$
Adding: $(3 - 0.2 \times 5) \times 9.8 = 8a$, so $a = 2.45$
$v^2 = 2 \times 2.45 \times 1.6 = 7.84$, so $v = 2.8$

After B lands, A decelerates at $\mu g = 1.96$: extra distance $= \frac{2.8^2}{2 \times 1.96} = 2$

Total distance = 1.6 + 2
Answer: 3.6 m

7 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 12 pairs of values: $\Sigma x = 84$, $\Sigma y = 708$, $\Sigma x^2 = 734$, $\Sigma xy = 5175$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 3 and 12. Estimate y when x = 11 and comment on its reliability.

$S_{xx} = 734 - \frac{84^2}{12} = 146$

$S_{xy} = 5175 - \frac{84 \times 708}{12} = 219$

$b = \frac{S_{xy}}{S_{xx}} = \frac{219}{146} = \frac{3}{2}$

$a = \bar{y} - b\,\bar{x} = 59 - (\frac{3}{2})(7) = 48.5$

$y = 48.5 + 1.5x$
At x = 11: y = 65. This is extrapolation (outside the data range), so may be unreliable
Answer: (a) $y = 48.5 + 1.5x$ (b) 65, unreliable (extrapolation)

8 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 25\,e^{-x}$ meet at the point P.
Find the exact area of the region bounded by the two curves and the y-axis.

$e^x = 25\,e^{-x}$ gives $e^{2x} = 25$, so $e^x = 5$, $x = \ln 5$

Area $= \int_0^{\ln 5} (25\,e^{-x} - e^x)\,dx = [-25\,e^{-x} - e^x]$
$= (-5 - 5) - (-25 - 1)$
Answer: 16

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 16)$. H_0: $\mu = 65$ is tested against H_1: $\mu > 65$ using the mean of a sample of size 25 at the 10% level.
(a) Find the critical region for the sample mean.
(b) Find the probability of a Type II error when $\mu = 66.5$.

Under H_0, sample mean $\sim N(65, \frac{16}{25})$, standard error = 0.8000
$z = 1.2816$ for the upper tail: critical value $= 65 + 1.2816 \times 0.8000 = 66.03$
Type II error: do not reject H_0, sample mean < 66.03, where mean $\sim N(66.5, 0.8000)$
$z = \frac{66.03 - 66.5}{0.8000} = -0.593$
Probability = 0.276
Answer: (a) sample mean > 66.0 (b) 0.276

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}-5\\-1\end{pmatrix} + t\begin{pmatrix}-3\\1\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-3\\-7\end{pmatrix} + t\begin{pmatrix}-5\\2\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}2\\-6\end{pmatrix} + t\begin{pmatrix}-2\\1\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}-2\\1\end{pmatrix} = 0$

$-10 + t(5) = 0$, so $t = 2$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-2\\-4\end{pmatrix}$, with length $\sqrt{20}$

Answer: t = 2, minimum distance = $2\sqrt{5}$ km