Day 58 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces ($\mathbf{i} + 6\mathbf{j}$) N, ($8\mathbf{i} + k\mathbf{j}$) N and ($-3\mathbf{j}$) N act on a particle of mass 2 kg. The resultant is parallel to ($9\mathbf{i} - 4\mathbf{j}$). Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(9)\mathbf{i} + (3 + k)\mathbf{j}$

Parallel to ($9\mathbf{i} - 4\mathbf{j}$): $\frac{3 + k}{9} = \frac{-4}{9}$, so $k = -7$

R = ($9\mathbf{i} - 4\mathbf{j}$) N, so $|R| = \sqrt{(9^2 + -4^2)} = 9.849$

a = |R| ÷ 2

Answer: (a) $k = -7$ (b) 4.92 m s^{-2}

2 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 30 observations x, $\Sigma(x - 40) = 120$ and $\Sigma(x - 40)^2 = 930$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 40) = \frac{120}{30} = 4$, so mean of $x = 44$

Variance = $\frac{930}{30} - (4)^2 = 15$

$\sigma = \sqrt{15}$

$\Sigma x^2 = \Sigma(x - 40)^2 + 80\,\Sigma(x - 40) + 30 \times 40^2$

$= 930 + 9600 + 48000$

Answer: (a) 44 (b) 3.87 (c) 58530

3 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_1^e \frac{1}{x(\ln x + 1)}\,dx$$

Let $u = \ln x + 1$, $du = \frac{1}{x}dx$

Integral = $\int_1^2 \frac{1}{u}\,du = [\ln u]$

$= \ln 2 - \ln 1$

Answer: ln 2

4 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=6}^{13}(4 \times 2^r + 1r)$.

Geometric part: first term $4 \times 2^6 = 256$, ratio 2, 8 terms.

Sum = $\frac{256(2^8 - 1)}{1} = 65280$.

Arithmetic part: 1 × 8 terms, average (6 + 13)/2: sum = 76.

Total = 65280 + 76.

Answer: 65356

Day 58 **A Level Maths: S Tier 10-a-day: Model Solutions**

5 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £1. Two fair dice are rolled: a total of at least 12 wins £14, a total of at most 3 wins £2, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 12) = $\frac{1}{36}$, P(total ≤ 3) = $\frac{1}{12}$, otherwise $\frac{8}{9}$

Profit: £13 with $\frac{1}{36}$, £1 with $\frac{1}{12}$, £-1 with $\frac{8}{9}$

(a) E = $-\frac{4}{9}$

(b) E(P^2) = $\frac{17}{3}$, Var = $\frac{17}{3} - (-\frac{4}{9})^2 = \frac{443}{81}$, so $\sigma = 2.339$

(c) Fair when the stake equals E(winnings) = $\frac{1}{36} \times 14 + \frac{1}{12} \times 2 = \frac{5}{9}$

Answer: (a) $-£\frac{4}{9}$ (b) 2.34 (c) $£\frac{5}{9}$

6 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = x + 9 to find the exact value of

$$\int_0^{16} \frac{x}{\sqrt{x+9}}\,dx$$

x = u $-$ 9, dx = du; limits become u = 9 and u = 25

Integral = $\int_9^{25} \frac{u-9}{\sqrt{u}}\,du = [\frac{2}{3}u^{3/2} - 18u^{1/2}]$

Evaluate the limits

Answer: $\frac{88}{3}$

7 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

f(x) = $\frac{-11x+3}{(1-3x)(1-x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $-11x + 3 \equiv A(1-x) + B(1-3x)$: A = -1, B = 4.

f(x) = $\frac{-1}{1-3x} + \frac{4}{1-x}$.

(b) $\frac{1}{1+cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use c = -3, then c = -1).

Coefficients of x^j: $-1(3)^j + 4(1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) A = -1, B = 4 (b) $3 + x - 5x^2 - 23x^3 + \ldots$; $|x| < \frac{1}{3}$

8 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 6 m and weight 200 N has A on rough ground ($\mu = 0.4$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = \frac{4}{3}$.

A man of weight 800 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2}\cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, $R = W$: need $\frac{W}{2\tan\theta} \le \mu W$, so $\mu \ge \frac{1}{2 \times \frac{4}{3}} = 0.375$

With the man at distance x: $R = 1000$, and moments about A give $S L \sin\theta = (200 \times 3 + 800x)\cos\theta$

Limiting: $S = \mu R = 0.4 \times 1000 = 400$, so $400 \times 6 \times \frac{4}{3} = 200 \times 3 + 800x$

$800x = 3200 - 600$, so $x = 3.25$

Answer: 3.25 m

9 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of $\tan^{-1} 2 + \tan^{-1} 3$,
giving your answer as a multiple of π.

$\tan(\tan^{-1}2 + \tan^{-1}3) = \frac{2+3}{1-6} = -1$

Both angles are acute with $\tan^{-1}2 > \pi/4$ and $\tan^{-1}3 > \pi/4$, so the sum is between $\pi/2$ and π
Hence the sum is $\pi - \pi/4$

Answer: $\frac{3\pi}{4}$

10 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = 3(t + \frac{1}{t})$, $y = (t - \frac{1}{t})$, $t \neq 0$.

Find a Cartesian equation of the curve.

$x^2 = 3^2(t^2 + 2 + \frac{1}{t^2})$.

$y^2 = (t^2 - 2 + \frac{1}{t^2})$.

$\frac{x^2}{9} - \frac{y^2}{1} = 4$.

Answer: $\frac{x^2}{9} - \frac{y^2}{1} = 4$