Day 57 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 50 and second term 8. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{8}{r}$. $S_\infty = \frac{a}{1-r} = \frac{8}{r(1-r)} = 50$.

$50r^2 - 50r + 8 = 0$.

$(5r - 1)(5r - 4) = 0$: $r = \frac{1}{5}$ or $r = \frac{4}{5}$.

Need $|r| < 1$: $\frac{1}{5}$, $\frac{4}{5}$ valid.

Answer: $r = \frac{1}{5}$, $a = 40$ or $r = \frac{4}{5}$, $a = 10$

2 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 40 observations x, $\Sigma(x - 30) = 20$ and $\Sigma(x - 30)^2 = 1090$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 30) = \frac{20}{40} = 0.5$, so mean of $x = 30.5$

Variance $= \frac{1090}{40} - (0.5)^2 = 27$

$\sigma = \sqrt{27}$

$\Sigma x^2 = \Sigma(x - 30)^2 + 60\,\Sigma(x - 30) + 40 \times 30^2$

$= 1090 + 1200 + 36000$

Answer: (a) 30.5 (b) 5.20 (c) 38290

3 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 20 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 41,
(b) the least number of rows for at least 1200 seats.

Arithmetic: $a = 20$, $d = 3$.

(a) $u_{41} = 20 + 40 \times 3 = 140$.

(b) $S_n = \frac{n}{2}(40 + 3n - 3) = \frac{n(3n + 37)}{2} \geq 1200$.

$S_{22} = 1133 < 1200$, $S_{23} = 1219 \geq 1200$.

Answer: (a) 140 (b) 23 rows

4 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $6\cos\theta - 8\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $6\cos\theta - 8\sin\theta = -3$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{6^2 + 8^2} = 10$ and $\tan\alpha = \frac{8}{6}$, so $\alpha = 53.1°$

So $6\cos\theta - 8\sin\theta \equiv 10\cos(\theta + 53.1°)$

$\cos(\theta + \alpha) = \frac{-3}{10}$

$\theta + \alpha = \pm 107.5°$ (using $\alpha = 53.1°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) $R = 10$, $\alpha = 53.1°$ (b) $\theta = 54.3°, 199.4°$

5 Grouped median with an unknown frequency (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The table shows the times, t minutes, taken by some runners.

Using linear interpolation, the median is estimated as 24 minutes. Find the value of k.

Total = 30 + k, so the median class is 20 to 30

Median = $20 + \dfrac{\frac{(30+k)}{2} - (7+k)}{5} \times 10 = 24$

Simplify: $8 - k/2 = 2$

$k = 12$

Answer: k = 12

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = \dfrac{\ln x}{x^2}$ for $x > 0$,

and determine its nature.

$dy/dx = \dfrac{1 - 2\ln x}{x^3} = 0$ gives $\ln x = \dfrac{1}{2}$

$y = \dfrac{1}{2}e^{-1}$ when $x = \sqrt{e}$

Second derivative at this point is negative, so it is a maximum

Answer: $(\sqrt{e}, \frac{1}{2}e^{-1})$, maximum

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(2r - 4) = 868$, find n.

$\sum_{r=1}^{n}(2r - 4) = 2 \times \dfrac{n(n+1)}{2} - 4n.$

$\dfrac{2\,n(n+1)}{2} - 4n = 868$, so $2n^2 - 6n - 1736 = 0$.

Solving ($n > 0$): $n = 31$.

Answer: n = 31

8 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (3 m, 100 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{5}{12}$, where α is the angle between string and rod. A 40 N particle hangs at P, AP = 0.75 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Moments about A: $T\sin\alpha \times 3 = 100 \times 1.5 + 40 \times 0.75$

$T \times \frac{5}{13} \times 3 = 180$, so $T = 156$

Horizontal: $H = T\cos\alpha = 144$. Vertical: $V + T\sin\alpha = 100 + 40$, so $V = 80$

Force $= \sqrt{(144^2 + 80^2)} = 165$, angle $= \tan^{-1}(\frac{80}{144}) = 29.1°$ above the horizontal

Answer: (a) 156 N (b) 165 N at 29.1° above the horizontal (away from the wall)

9 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \frac{1}{\sqrt{x+2}}$, the x-axis and the lines $x = 0$ and $x = 4$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^4 y^2\,dx = \pi\int_0^4 \left(\frac{1}{\sqrt{x+2}}\right)^2 dx$

Simplify the integrand, then integrate

$V = \pi \times (\ln 3)$

Answer: $\pi\ln 3$

10 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 105 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.5$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 105\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 105\sin\alpha$, so $H(\frac{3}{5} + 0.5 \times \frac{4}{5}) = 105(\frac{4}{5} - 0.5 \times \frac{3}{5})$

$H_{min} = 52.5$

About to slide up: $H\cos\alpha = 105\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.5 \times \frac{4}{5}) = 105(\frac{4}{5} + 0.5 \times \frac{3}{5})$

$H_{max} = 577.5$

Answer: $52.5 \leq H \leq 577.5$