Day 56 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 58.8 m above the ground for 4 s.
Find the speed of projection and the greatest height reached. Take g = 9.8 m s^{-2}.

By symmetry, the ball takes 2 s to rise from height 58.8 m to its highest point

Speed at height 58.8 m: $v = 9.8 \times 2 = 19.6$ m s^{-1}

Speed of projection: $u^2 = v^2 + 2gh = 384.16 + 2 \times 9.8 \times 58.8 = 1536.64$

$u = 39.2$. Greatest height = $\frac{u^2}{(2g)} = \frac{1536.64}{19.6} = 78.4$

Answer: 39.2 m s^{-1}, 78.4 m

2 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (0, 2) and (4, 512).
Find the exact values of a and b, and the value of y when x = 6.

$ab^0 = 2$ and $ab^4 = 512$

Divide: $b^4 = 256$ so $b = 4$

$a = 2 \div 4^0 = 2$

$y = 2 \times 4^6$

Answer: a = 2, b = 4, y = 8192

3 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 + 5x - 6 \leq 0$ and $2x + 7 \leq 6$.

Quadratic: $(x + 6)(x - 1) \leq 0$ gives $-6 \leq x \leq 1$.

Linear: $2x + 7 \leq 6$ gives $x \leq -\frac{1}{2}$.

Both must hold: take the overlap of the two sets.

Answer: $-6 \leq x \leq -\frac{1}{2}$

4 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 2, 3, 4, 5 with $P(X = x) = k(6 - x)$. X_1 and X_2 are independent copies of X.
(a) Find k.
(b) Find $P(X_1 + X_2 = 6)$.
(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(4 + 3 + 2 + 1) = 10k = 1$, so $k = \frac{1}{10}$

(b) Ways to total 6: (2,4), (3,3), (4,2)

$P = \frac{2}{5} \times \frac{1}{5} + \frac{3}{10} \times \frac{3}{10} + \frac{1}{5} \times \frac{2}{5} = \frac{1}{4}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{3}{10}$

$E(X) = 3$, so $E(X_1 + X_2) = 2 \times 3 = 6$

Answer: (a) $k = \frac{1}{10}$ (b) $\frac{1}{4}$ (c) $\frac{3}{10}$, 6

5 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $3\sec^2\theta + 11\tan\theta + 3 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $3\tan^2\theta + 11\tan\theta + 6 = 0$
$(\tan\theta + 3)(3\tan\theta + 2) = 0$

$\tan\theta = -3$ or $\tan\theta = -\frac{2}{3}$

$\theta = 108.4°$ and $146.3°$ (+180°)

Answer: θ = 108.4°, 146.3°, 288.4°, 326.3°

6 Slide down a rough slope then along rough ground (CIE)

A particle starts from rest at the top of a rough slope of length 5 m ($\sin\alpha = \frac{4}{5}$, $\mu = 0.25$), then slides along rough horizontal ground with the same μ.

Use energy to find (a) its speed at the foot of the slope (3 s.f.), (b) the distance it slides on the ground. Take $g = 9.8$ m s^{-2}.

On the slope: $\frac{1}{2} m v^2 = m g \times 5 \sin\alpha - \mu m g \cos\alpha \times 5$

$v^2 = 2 \times 9.8 \times 5(\frac{4}{5} - 0.25 \times \frac{3}{5}) = 63.7$, so $v = 7.981$

On the ground: $\frac{1}{2} m v^2 = \mu m g x$, so $x = \frac{v^2}{2\mu g} = \frac{63.7}{4.9} = 13$

Answer: (a) 7.98 m s^{-1} (b) 13 m

7 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 - 7x^2 + 13x - 8}{(x-2)(x-3)}$ in the form $ax + b + \frac{C}{(x-2)} + \frac{D}{(x-3)}$.

Divide: the numerator has degree 3, the denominator degree 2.

$(x^3 - 7x^2 + 13x - 8) \div (x^2 - 5x + 6)$ gives quotient $x - 2$.

Remainder: $-3x + 4$.

Put $x = 2$: remainder $= C(-1)$ so $C = 2$. Put $x = 3$: $D = -5$.

Answer: $x - 2 + \frac{2}{(x-2)} - \frac{5}{(x-3)}$

8 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-x^2 - x}{(x-4)(x-2)^2}$ in the form

$\frac{A}{(x-4)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$.

$-x^2 - x \equiv A(x-2)^2 + B(x-4)(x-2) + C(x-4)$.

Put $x = 4$: $-20 = A(2)^2$, so $A = -5$.

Put $x = 2$: $-6 = C(-2)$, so $C = 3$.

Compare x^2: $-1 = 1A + 1B$, so $B = 4$.

Answer: $A = -5$, $B = 4$, $C = 3$

9 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a W, P(first was a W) $= \frac{1}{3}$. Find x.

P(W then W) $= 0.2x$, P(second W) $= 0.2x + 0.6(1 - x)$

P(first W | second W) $= \frac{0.2x}{0.2x + 0.6(1-x)} = \frac{1}{3}$

$0.2x = \frac{1}{3}(0.2x + 0.6 - 0.6x)$

Solve the linear equation for x: $x = \frac{3}{5}$

Answer: $x = \frac{3}{5}$

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -8 \\ 8 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$.

The point P has coordinates $(-3, 0, 3)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 5 \\ -8 \\ -5 \end{pmatrix} - t\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$

FP · d = 0: $36 - t(12) = 0$, so $t = 3$

F = $(-2, 2, 2)$

$\mathrm{PF} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{6}$

Answer: F = $(-2, 2, 2)$, PF = $\sqrt{6}$