Day 54 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $5k = n^2 + 2$ has no integer solutions (k and n integers).

Every integer is 5q, 5q + 1, 5q + 2, 5q + 3, 5q + 4 for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 5$, so:

$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 4$; $4 \to 1$ ($r \to n^2 \bmod 5$)

So $n^2 + 2 \equiv 2$ or 3 or 1 (mod 5).

No residue is 0, so $n^2 + 2$ is never a multiple of 5.

Answer: Proved

2 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 3π cm^{3}s^{-1}. Find the rate at which the depth h is rising when h = 2 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $3\pi = \frac{\pi}{4} \times 4 \times dh/dt$

$dh/dt = \frac{12}{4}$

Answer: 3 cm s^{-1}

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 73.5 m above the ground for 2 s. Find the speed of projection and the greatest height reached. Take g = 9.8 m s^{-2}.

By symmetry, the ball takes 1 s to rise from height 73.5 m to its highest point

Speed at height 73.5 m: $v = 9.8 \times 1 = 9.8$ m s^{-1}

Speed of projection: $u^2 = v^2 + 2gh = 96.04 + 2 \times 9.8 \times 73.5 = 1536.64$

$u = 39.2$. Greatest height = $\frac{u^2}{(2g)} = \frac{1536.64}{19.6} = 78.4$

Answer: 39.2 m s^{-1}, 78.4 m

4 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0 e^{kt}$, t in hours. Initially P = 400 and after 4 hours P = 1200. Find k, and the time for the population to reach 2400, to 3 s.f.

$P_0 = 400$. $400\, e^{4k} = 1200$ so $e^{4k} = 3$

$k = \frac{1}{4} \ln 3 = 0.275$

$400\, e^{kt} = 2400$ so $e^{kt} = 6$, $t = \frac{1}{k} \ln 6$

t = 6.52 hours

Answer: k = 0.275, t = 6.52 hours

5 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £1000 into an account on 1 January. Interest of 3% is added each 31 December. Find the first year-end at which the balance exceeds £8,000.

After n years: $1000(1.03) + 1000(1.03^2) + \ldots + 1000(1.03^n)$.

GP: a = 1000 × 1.03, r = 1.03: $S_n = \frac{1000 \times 1.03\,(1.03^n - 1)}{0.03}$.

$S_n > 8000 \Rightarrow 1.03^n > 1.2330 \Rightarrow n > 7.086$.

So n = 8 years.

Answer: after 8 years

6 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each battery is faulty with probability 0.2, independently.
Find the least number n of batterys that must be tested so that the probability of finding at least two faulty is greater than 0.99.

Let $X \sim B(n, 0.2)$. $P(X \geq 2) = 1 - 0.8^n - n(0.2)(0.8)^{n-1}$
This has no simple algebraic solution, so test values of n
n = 30: $P(X \geq 2) = 0.9895$, which is not greater than 0.99
n = 31: $P(X \geq 2) = 0.9913$, which is greater than 0.99
$P(X \geq 2)$ increases with n, so the least value is 31
Answer: n = 31

7 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 9 kg and 3 kg hang on a light string over a smooth fixed pulley. The 9 kg particle is 0.5 m above the ground; the 3 kg one is on the ground.
Released from rest, find (a) the tension, (b) the greatest height reached by the 3 kg particle. Take $g = 9.8$ m s^{-2}.

$9g - T = 9a$ and $T - 3g = 3a$, so $a = \frac{9-3}{9+3} \times 9.8 = 4.9$

$T = 9(9.8 - 4.9) = 44.1$

The 3 kg particle rises 0.5 m before the 9 kg particle lands, reaching $v^2 = 2 \times 4.9 \times 0.5 = 4.9$

The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{4.9}{19.6} = 0.25$

Greatest height = 0.5 + 0.25
Answer: (a) 44.1 N (b) 0.75 m

8 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 10 pairs of values: $\Sigma x = 110$, $\Sigma y = 350$, $\Sigma x^2 = 1365$, $\Sigma xy = 3726$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 7 and 16. Estimate y when x = 17 and comment on its reliability.

$S_{xx} = 1365 - \frac{110^2}{10} = 155$

$S_{xy} = 3726 - \frac{110 \times 350}{10} = -124$

$b = \frac{S_{xy}}{S_{xx}} = \frac{-124}{155} = -\frac{4}{5}$

$a = \bar{y} - b\,\bar{x} = 35 - (-\frac{4}{5})(11) = 43.8$

$y = 43.8 - 0.8x$
At x = 17: y = 30.2. This is extrapolation (outside the data range), so may be unreliable
Answer: (a) $y = 43.8 - 0.8x$ (b) 30.2, unreliable (extrapolation)

9 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{x+2}{-x+4} > -2$.

Rearrange: $\frac{x+2}{-x+4} - (-2) > 0$.

Combine: $\frac{-x+10}{-x+4} > 0$.

Critical values: x = 10 (zero of numerator), x = 4 (excluded, denominator zero).
Sign analysis of the three regions gives the solution.
Answer: $x < 4$ or $x > 10$

10 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $8\sec^2\theta + 10\tan\theta - 6 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $8\tan^2\theta + 10\tan\theta + 2 = 0$
$2(4\tan\theta + 1)(\tan\theta + 1) = 0$

$\tan\theta = -\frac{1}{4}$ or $\tan\theta = -1$

$\theta = 166.0°$ and $135.0°$ $(+180°)$
Answer: θ = 135.0°, 166.0°, 315.0°, 346.0°