Day 53 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 2x + 9$ for $x \geq k$. Given that k is the least value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = (x - 1)^2 + 8$.
The vertex is at $x = 1$, so $k = 1$ (f is one-one on one side of it).
Let $y = f(x)$: $y - 8 = (x - 1)^2$, so $(x - 1) = \pm\sqrt{y - 8}$.
As $x \geq 1$, take the positive root: $x = 1 + \sqrt{y - 8}$.
Range of f = domain of f^{-1}: $f(1) = 8$, so $x \geq 8$.
Answer: $k = 1$; $f^{-1}(x) = 1 + \sqrt{x - 8}$; $x \geq 8$

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sin(2\theta + 30°) = \frac{1}{2}$ for $0° \leq \theta \leq 360°$.

Let $u = 2\theta + 30°$; then $\sin u = \frac{1}{2}$
Base solutions: $u = 150°, 30°$ (+ multiples of 360°)
$\theta = (u - 30°) \div 2$, keeping those in range
Answer: $\theta = 0°, 60°, 180°, 240°, 360°$

3 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = \frac{1}{2}$ and $u_{n+1} = \frac{1}{1 - u_n}$.
Show that the sequence is periodic and find $\sum_{r=1}^{75} u_r$.

$u_1 = \frac{1}{2}$, $u_2 = 2$, $u_3 = -1$, $u_4 = \frac{1}{2}$.
$u_4 = u_1$, so the sequence repeats with period 3.
One period sums to $\frac{3}{2}$.
$75 = 3 \times 25 + 0$: 25 full periods plus the first 0 terms.
Answer: $\frac{75}{2}$

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 13 and upper quartile 21. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 12 observations have mean 13. The largest is 35, an outlier, and it is removed. Find the new mean.

IQR = 21 − 13 = 8, so 1.5 × IQR = 12
Lower limit = 13 − 12 = 1, upper limit = 21 + 12 = 33
$\Sigma x = 12 \times 13 = 156$, so without 35: $\Sigma x = 121$
New mean = $\frac{121}{11}$
Answer: (a) 1 to 33 (b) 11.00

5 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - xy + 2y^2 = 53$.
Find the equation of the normal to C at the point $(-3, 4)$,
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At $(-3, 4)$: $dy/dx = \frac{10}{19}$

Normal gradient $= -1 \div \frac{10}{19} = -\frac{19}{10}$

$y - 4 = (-\frac{19}{10})(x - -3)$, then clear fractions

Answer: 19x + 10y + 17 = 0

6 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from level ground with horizontal and vertical velocity components 44.1 and 68.6 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 44.1$
Moving upwards: $68.6 - 9.8t = 44.1$, so $t = 2.5$
Moving downwards: $68.6 - 9.8t = -44.1$, so $t = 11.5$
Horizontal distance $= 44.1 \times (11.5 - 2.5)$
Answer: (a) 2.5 s and 11.5 s (b) 396.9 m

7 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{-3x - 11}{x + 1}$.
(a) Write down the equations of the asymptotes of C.
(b) The line $y = 2x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = -1$. As $x \to \infty$, $y \to -3$, so $y = -3$.
(b) $-3x - 11 = (2x + k)(x + 1)$.
$2x^2 + (k + 5)x + (k + 11) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 + 2k - 63 = 0$.
Answer: (a) x = −1, y = −3 (b) k = −9 or k = 7

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (1, 2, 4), (−2, −1, −2), (−3, −2, −2).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$, $BC = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$; $BA \cdot BC = -6$

$|BA| = \sqrt{54}$, $|BC| = \sqrt{2}$

$\cos ABC = \frac{-6}{\sqrt{54}\sqrt{2}} = -0.5774$, so ABC = 125.3°

$|BA \times BC| = \left|\begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix}\right| = \sqrt{72}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 125.3°, area = $3\sqrt{2}$

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 3 kg on it ($\mu = 0.2$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 47.04. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The block is accelerated only by friction: $F = 3a \leq \mu mg$

So $a \leq \mu g = 1.96$, and P_max = 8 × 1.96 = 15.68

P = 47.04 > 15.68, so the block slips. Block: $a = \mu g = 1.96$

Plank: $47.04 - 5.88 = 5a$, so $a = 8.232$

Answer: (a) 15.68 N (b) block 1.96 m s^{-2}, plank 8.232 m s^{-2}

10 Grouped median with an unknown frequency (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The table shows the times, t minutes, taken by some runners.

Using linear interpolation, the median is estimated as 11.5 minutes. Find the value of k.

Total = 39 + k, so the median class is 10 to 15

$$\text{Median} = 10 + \frac{\frac{(39+k)}{2} - (13+k)}{5} \times 5 = 11.5$$

Simplify: $6.5 - k/2 = 1.5$

k = 10

Answer: k = 10