Day 52 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$, solve
$\sin 3\theta = 2\sin\theta$ for $0 < \theta < \pi$, in terms of π.

Substitute: $\sin\theta(3 - 4\sin^2\theta - 2) = 0$

$\sin\theta = 0$ or $\sin^2\theta = \frac{1}{4}$

$\sin\theta = 0$ gives no solutions in the range

$\sin^2\theta = \frac{1}{4}$ gives $\frac{\pi}{6}, \frac{5\pi}{6}$

Answer: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

2 Poisson: scaling the rate, then binomial (CIE)

Customers arrive at a till at an average rate of 4 per hour, independently and at random.
(a) Find the probability of at least 2 in a 20-minute period.
(b) Find the probability that, of 9 separate 20-minute periods, exactly 2 have no arrivals.

20-minute period: $\lambda = 4 \times \frac{20}{60} = \frac{4}{3}$, so $X \sim Po(\frac{4}{3})$

(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.3849$

(b) $P(X = 0) = e^{-\frac{4}{3}} = 0.2636$
$Y \sim B(9, 0.2636)$: $P(Y = 2) = 0.294$
Answer: (a) 0.385 (b) 0.294

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 4 strips to estimate

$\int_0^1 \ln(1 + x^2)\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{4}$; ordinates at $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots, 1$

Sum $= \frac{h}{2}[y_0 + y_4 + 2(y_1 + \ldots + y_3)] = 0.2692$

$f''(x) > 0$ on the interval, so the curve is convex and the trapezia lie above the curve
Answer: 0.2692; overestimate

4 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{4}{5}u_n + 4$, $u_1 = 32$.

(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{4}{5}L + 4$, so $L(1 - \frac{4}{5}) = 4$.

$L = 20$.

(b) $u_n - L = (u_1 - L)(\frac{4}{5})^{n-1} = 12(\frac{4}{5})^{n-1}$.

Need $12 \times (\frac{4}{5})^{n-1} < 0.001$, so $n - 1 > 42.092$.

The least n is 44.
Answer: (a) L = 20 (b) n = 44

5 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 10 m and mass 30 kg rests on supports at C and D, where AC = 2 m and DB = 3 m. A girl of mass 50 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(2) = 50(x - 7)$, so $x = 7 + \frac{30 \times 2}{50} = 8.2$

Tilting about C (girl before C): $30(3) = 50(2 - x)$, so $x = 2 - \frac{30 \times 3}{50} = 0.2$

Answer: $0.2 \leq x \leq 8.2$

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x^4e^{-2x}$ for $x > 0$,
and determine its nature.

$dy/dx = x^3e^{-2x}(4 - 2x) = 0$ gives $x = 0$ or $x = 2$
$y = 16e^{-4}$ when $x = 2$
Second derivative at this point is negative, so it is a maximum
Answer: $(2, 16e^{-4})$, maximum

7 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{5x - 1} = x - 3$.

Square both sides: $5x - 1 = (x - 3)^2$.
$x^2 - 11x + 10 = 0$, so $x = 1$ or $x = 10$.
Check in $\sqrt{5x - 1} = x - 3$: $x = 10$ gives $7 \geq 0$ (valid).
$x = 1$ gives right-hand side $-2 < 0$, so it is rejected.
Answer: $x = 10$

8 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−7, 4) and B(−8, −3). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-7))^2 + 16 = (c - (-8))^2 + 9$.
Expand and solve: $-2c = 8$, so $c = -4$.
Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$
$= \frac{1}{2}|(-3) \times (-3) - (-4) \times 4| = \frac{25}{2}$.

Answer: C(−4, 0); area $\frac{25}{2}$

9 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where
$v = 36\sqrt{t} - 6t$.
Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{36}{2\sqrt{t}} - 6 = 0$ when $\sqrt{t} = 3$, so $t = 9$
$v = 36 \times 3 - 6 \times 9 = 54$
$v = 0$ when $\sqrt{t}(36 - 6\sqrt{t}) = 0$: $t = 36$ ($t = 0$ is the start)
$s = \int v\,dt = \frac{2}{3} \times 36\, t^{\frac{3}{2}} - \frac{6}{2}t^2$, so at $t = 36$: $s = 5184 - 3888$

Answer: (a) 54 m s^{-1} (b) 1296 m

10 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(200, 0.25).

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate P(38 ≤ X ≤ 56), to 3 significant figures.

np = 50 and nq = 150, both greater than 5 (n large, p not extreme)

Y ~ N(50, 37.5) with $\sigma = 6.124$

Apply the continuity correction:

P(38 ≤ X ≤ 56) ≈ P(37.5 < Y < 56.5) = 0.8558 − 0.02061 = 0.835

Answer: 0.835