Day 51 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at x = 2 and touches at x = −2, so $f(x) = k(x - 2)(x + 2)^2$.

At (0, 16): $16 = k(-2)(2)^2 = -8k$, so k = −2.

$f(x) = -2(x - 2)(x + 2)^2$, then expand.

Answer: $f(x) = -2x^3 - 4x^2 + 8x + 16$

2 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 8 strips to estimate

$\int_1^2 \sqrt{x}e^x \, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{8}$; ordinates at x = 1, $\frac{9}{8}$, $\frac{5}{4}$, $\frac{11}{8}$, …, 2

Sum = $\frac{h}{2}[y_0 + y_8 + 2(y_1 + \ldots + y_7)] = 5.862$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 5.862; overestimate

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(4\mathbf{i} - 5\mathbf{j})$ N, $(7\mathbf{i} + k\mathbf{j})$ N and $(\mathbf{i} + 2\mathbf{j})$ N act on a particle of mass 2 kg. The resultant is parallel to $(6\mathbf{i} - \mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(12)\mathbf{i} + (-3 + k)\mathbf{j}$

Parallel to $(6\mathbf{i} - \mathbf{j})$: $\frac{-3 + k}{12} = \frac{-1}{6}$, so k = 1

R = $(12\mathbf{i} - 2\mathbf{j})$ N, so $|R| = \sqrt{(12^2 + -2^2)} = 12.17$

a = |R| ÷ 2

Answer: (a) k = 1 (b) 6.08 m s^{-2}

4 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 8 and variance 4.8.

(a) Find n and p.

(b) Find P(X = 8), correct to 3 significant figures.

np = 8 and np(1 − p) = 4.8

Divide: $1 - p = 4.8 \div 8 = \frac{3}{5}$, so $p = \frac{2}{5}$

$n = 8 \div \frac{2}{5} = 20$

$P(X = 8) = C(20, 8) (\frac{2}{5})^8 (\frac{3}{5})^{12} = 0.180$

Answer: (a) n = 20, p = $\frac{2}{5}$ (b) 0.180

5 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 - n + 11$ is prime for every positive integer n.

Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.

For n = 11: $11^2 - 11 + 11 = 121$.

121 = 11 × 11, which is not prime.

All smaller values of n give a prime, so n = 11 is the smallest.

Answer: n = 11 (value 121 = 11 × 11)

6 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 2 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 2 m s^{-2} to rest.
The journey takes 38 s and covers 522 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{2}$, time decelerating = $\frac{V}{2}$, so time at constant speed = $38 - \frac{V}{2} - \frac{V}{2}$

Area under the speed-time graph: $522 = 38V - \frac{V^2}{4} - \frac{V^2}{4}$

Multiply by 2ab: $4V^2 - 304V + 4176 = 0$
V = 18 or V = 58
V = 58 gives time at constant speed −20 s, which is negative, so V = 18
Time at constant speed = 38 − 9 − 9 = 20 s
Answer: V = 18, 20 s

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve $y = \frac{15}{x}$ and the line y = 8 − x.

Find the exact area of the shaded region enclosed by the curve and the line.

Intersect: $\frac{15}{x} = 8 - x$ gives $x^2 - 8x + 15 = 0$, so x = 3 and x = 5

Area = $\int_3^5 (8 - x - \frac{15}{x})\,dx = [8x - \frac{x^2}{2} - 15 \ln x]$

$= (40 - \frac{25}{2} - 15 \ln 5) - (24 - \frac{9}{2} - 15 \ln 3)$

Answer: $8 + 15\ln 3 - 15\ln 5$

8 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{3}{-2-7} = -\frac{1}{3}$.
Gradient of l_2 = 3. Equation: y − 3 = 3(x + 2).
y = 0: x = −3, so C(−3, 0).
AB = $3\sqrt{10}$, BC = $\sqrt{10}$: area = ½ × AB × BC.
Answer: C(−3, 0); area 15

9 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix}-1\\-1\\2\end{pmatrix} + s\begin{pmatrix}0\\2\\1\end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix}-4\\1\\0\end{pmatrix} + t\begin{pmatrix}3\\0\\3\end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: −1 = −4 + 3t; −1 + 2s = 1; 2 + s = 3t
Solving the first two: s = 1, t = 1; the third equation is satisfied
Intersection: (−1, 1, 3)

$\cos\varphi = \frac{|3|}{\sqrt{5}\sqrt{18}} = 0.3162$, φ = 71.6°

Answer: Lines meet at (−1, 1, 3); angle 71.6°

10 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A student's time on a task, in minutes, is N(31, 9^2).
Given that it exceeds 40, find the probability that it exceeds 53.

Required = $P(X > 53 \mid X > 40) = \frac{P(X > 53)}{P(X > 40)}$

$P(X > 53) = P(Z > 2.444) = 0.007254$

$P(X > 40) = P(Z > 1.000) = 0.1587$

$= \frac{0.007254}{0.1587} = 0.0457$

Answer: 0.0457