**Day 5** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the top of a vertical cliff, 58.8 m high, at 24.5 m $s^{-1}$ at angle $\alpha$ above the horizontal ($\tan\alpha = \frac{4}{3}$), and lands on horizontal ground.
Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m $s^{-2}$.

Vertically (up positive): $-58.8 = 19.6\,t - 4.9t^2$, so $4.9t^2 - 19.6t - 58.8 = 0$
t = 6 s (the positive root)
Horizontal distance = 14.7 × 6 = 88.2 m
On landing: $v_x = 14.7$, $v_y = 19.6 - 9.8 \times 6 = -39.2$
speed = $\sqrt{(14.7^2 + 39.2^2)} = 41.9$, $\tan$ angle = $\frac{39.2}{14.7}$, angle = 69.4° below the horizontal

**Answer: (a) 6 s, 88.2 m (b) 41.9 m $s^{-1}$ at 69.4° below the horizontal**

## 2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0e^{-kt}$ grams, t in days.
Initially m = 160 and after 4 days m = 80.
Find k, and the time for the mass to fall to 20 grams, to 3 s.f.

$m_0 = 160$. $160\,e^{-4k} = 80$ so $e^{-4k} = \frac{1}{2}$

$k = \frac{1}{4}\ln 2 = 0.173$

$160\,e^{-kt} = 20$ so $e^{kt} = 8$, $t = \frac{1}{k}\ln 8$

t = 12.0 days

**Answer: k = 0.173, t = 12.0 days**

## 3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A javelin is projected from a point O on horizontal ground with speed 24.5 m $s^{-1}$ at an angle $\theta$ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m $s^{-2}$.

Vertical: $0 = 14.7^2 - 2 \times 9.8 \times H$, so H = 11.025
At height $\frac{H}{2} = 5.5125$: $v_y^2 = 14.7^2 - 2 \times 9.8 \times 5.5125 = 108.045$
$v_x = 19.6$ (constant), so speed$^2$ = 384.16 + 108.045 = 492.205

**Answer: (a) 11.025 m (b) 22.2 m $s^{-1}$**

## 4 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^3 + 8n$ is divisible by 3
for every positive integer n.

$n^3 + 8n = (n - 1)n(n + 1) + 9n$.
One of three consecutive integers is a multiple of 3,
so $(n - 1)n(n + 1)$ is divisible by 3.
9 is a multiple of 3, so 9n is too. Hence the sum is divisible by 3.

**Answer: Proved**

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## 5 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the stationary points of
$y = 3\cos 2x + 6\cos x$ for $0 < x < 2\pi$,
and determine their nature.

$dy/dx = -6\sin 2x - 6\sin x = -6\sin x(2\cos x + 1)$

$dy/dx = 0$: $\sin x = 0$ or $\cos x = -\frac{1}{2}$

$x = \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ in the range

Test the sign of $d^2y/dx^2$ at each

**Answer: $(\frac{2\pi}{3}, -\frac{9}{2})$ minimum; $(\pi, -3)$ maximum; $(\frac{4\pi}{3}, -\frac{9}{2})$ minimum**

## 6 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + ax^2 + bx + 60$, where a and b are constants.
$(x - 3)$ is a factor of $f(x)$, and $f(x)$ leaves remainder 18 when divided by $(x - 2)$.
Find a and b, and hence solve $f(x) = 0$.

$f(3) = 0$: $27 + 9a + 3b + 60 = 0$, so $9a + 3b + 87 = 0$.
$f(2) = 18$: $4a + 2b + 68 = 18$.
Solving: $a = -4$, $b = -17$.
$f(x) = x^3 - 4x^2 - 17x + 60 = (x + 4)(x - 3)(x - 5)$
**Answer: $a = -4$, $b = -17$; $x = -4, 3, 5$**

## 7 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each seed is faulty with probability 0.03, independently.
Find the least number n of seeds that must be tested so that the probability of finding at least one faulty is greater than 0.99.

$P(\text{at least one faulty}) = 1 - 0.97^n > 0.99$
$0.97^n < 0.01$
$n \ln 0.97 < \ln 0.01$
Dividing by $\ln 0.97$ (negative) reverses the inequality: $n > 151.2$
$n = 152$
**Answer: $n = 152$**

## 8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = 4 \sin t$, $y = 1 \cos 2t$.
Find y in terms of x.

$\cos 2t = 1 - 2\sin^2 t$.

$\sin t = \frac{x}{4}$.

$y = 1(1 - 2 \times \frac{x^2}{16}) = 1 - \frac{1}{8}x^2$.

**Answer: $y = 1 - \frac{1}{8}x^2$**

## 9 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$.

The point P has coordinates (6, 3, −5).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} - t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

FP · d = 0: 18 − t(18) = 0, so t = 1

F = (5, 4, −3)

$\mathrm{PF} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{6}$

**Answer: F = (5, 4, −3), PF = $\sqrt{6}$**

## 10 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of $H_0$: p = 0.4 against $H_1$: p > 0.4 uses a sample of 40 at the 5% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact p = 0.5.

X ~ B(40, 0.4). P(X ≥ 22) = 0.0392 and P(X ≥ 21) = 0.0744

The first is at most 0.05 and the second is not, so the critical region is X ≥ 22

Actual significance level = P(X ≥ 22) = 0.0392

Type II error: $H_0$ is not rejected, so X ≤ 21, with X ~ B(40, 0.5)

P(X ≤ 21) = 0.682

**Answer: (a) X ≥ 22 (b) 0.0392 (c) 0.682**