Day 49 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 3 m and DB = 2 m. A girl of mass 40 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(2) = 40(x - 6)$, so $x = 6 + \frac{30 \times 2}{40} = 7.5$

Tilting about C (girl before C): $30(1) = 40(3 - x)$, so $x = 3 - \frac{30 \times 1}{40} = 2.25$

Answer: $2.25 \leq x \leq 7.5$

2 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (1, 8) and (4, 512).

Find the exact values of a and b, and the value of y when x = 6.

$ab^1 = 8$ and $ab^4 = 512$

Divide: $b^3 = 64$ so $b = 4$

$a = 8 \div 4^1 = 2$

$y = 2 \times 4^6$

Answer: a = 2, b = 4, y = 8192

3 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 yellow counters and some grey counters, n in total.

Two are taken without replacement. P(both yellow) = $\frac{6}{55}$. Find n.

P(both yellow) = $\frac{4}{n} \times \frac{3}{(n-1)} = \frac{12}{n(n-1)}$

So $n(n-1) = 12 \div \frac{6}{55} = 110$

$n^2 - n - 110 = 0$, so $(n - 11)(n + 10) = 0$

$n > 0$, so $n = 11$

Answer: n = 11

4 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 - n + 5$ is prime for every positive integer n.

Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.

For n = 5: $5^2 - 5 + 5 = 25$.

$25 = 5 \times 5$, which is not prime.

All smaller values of n give a prime, so n = 5 is the smallest.

Answer: n = 5 (value 25 = 5 × 5)

5 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 5 kg is at rest on rough horizontal ground ($\mu = 0.1$). A horizontal force of 11.9 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8$ m s^{-2}.

Friction = $\mu mg = 0.1 \times 5 \times 9.8 = 4.9$ N

While pushed: $11.9 - 4.9 = 5a$, so $a = 1.4$

$v = 1.4 \times 7 = 9.8$; $s_1 = \frac{1}{2} \times 1.4 \times 7^2 = 34.3$

After release: deceleration = $\mu g = 0.98$, so $s_2 = \frac{9.8^2}{2 \times 0.98} = 49$

Total distance = 34.3 + 49

Answer: (a) 9.8 m s^{-1} (b) 83.3 m

6 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\sin(2\cos^{-1}(\frac{4}{5}))$.

Let A = $\cos^{-1}(\frac{4}{5})$, so $\cos A = \frac{4}{5}$ and $\sin A = \frac{3}{5}$

$\sin 2A = 2 \sin A \cos A = 2 \times \frac{3}{5} \times \frac{4}{5}$

Answer: $\frac{24}{25}$

7 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The sum of the first n terms of a sequence is

$S_n = 2n^2 + 6n$.

(a) Find an expression for u_n.

(b) Find the least n for which $u_n > 165$.

$u_n = S_n - S_{n-1} = 2n^2 + 6n - [2(n - 1)^2 + 6(n - 1)]$

$= 4n + 4$.

$u_1 = S_1 = 8$ also fits, so $u_n = 4n + 4$ for all n.

$4n + 4 > 165 \Rightarrow n > \frac{161}{4}$, so n = 41.

Answer: (a) $u_n = 4n + 4$ (b) n = 41

8 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$16^x \times 8^y = 1024$

$27^x \div 3^y = 3$

Write each side with prime bases: $1024 = 2^{10}$, $3 = 3^1$.

First: $2^{4x} \times 2^{3y} = 2^{10}$, so 4x + 3y = 10.

Second: $3^{3x - 1y} = 3^1$, so 3x − 1y = 1.

Solving the pair of linear equations: x = 1, y = 2.

Answer: x = 1, y = 2

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ N(μ, 100). H_0: μ = 71 is tested against H_1: μ < 71 using the mean of a sample of size 36 at the 5% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when μ = 69.5.

Under H_0, sample mean ~ N(71, $\frac{100}{36}$), standard error = 1.667

z = 1.6449 for the lower tail: critical value = 71 − 1.6449 × 1.667 = 68.26

Type II error: do not reject H_0, sample mean > 68.26, where mean ~ N(69.5, 1.667)

$z = \frac{68.26 - 69.5}{1.667} = -0.745$

Probability = 0.772

Answer: (a) sample mean < 68.3 (b) 0.772

10 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{2x} \cos x$. Show that $dy/dx = e^{2x}(2 \cos x - \sin x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (2)e^{2x} \cos x - e^{2x} \times \sin x$

dy/dx = 0 requires 2 cos x = 1 sin x

tan x = 2, smallest positive x = 1.1071

x = 1.11

Second derivative test (or gradient change): a maximum

Answer: x = 1.11, maximum