Day 48 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{2x + 1} = x - 7$.

Square both sides: $2x + 1 = (x - 7)^2$.

$x^2 - 16x + 48 = 0$, so $x = 4$ or $x = 12$.

Check in $\sqrt{2x + 1} = x - 7$: $x = 12$ gives $5 \geq 0$ (valid).

$x = 4$ gives right-hand side $-3 < 0$, so it is rejected.

Answer: $x = 12$

2 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 4 kg moves under a single force F N, where $\mathbf{F} = (32t - 12)\mathbf{i} + (24t^2 - 16)\mathbf{j}$ and t is the time in seconds. At t = 0, P is at the origin with velocity $(-3\mathbf{i} + 4\mathbf{j})$ m s^{-1}.

Find (a) the velocity of P at t = 3, (b) the position vector of P at t = 3.

$\mathbf{a} = \frac{\mathbf{F}}{4} = (8t - 3)\mathbf{i} + (6t^2 - 4)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\, dt = (4t^2 - 3t - 3)\mathbf{i} + (2t^3 - 4t + 4)\mathbf{j}$ (using $\mathbf{v}(0) = (-3\mathbf{i} + 4\mathbf{j})$)

At t = 3: $\mathbf{v} = (24\mathbf{i} + 46\mathbf{j})$

$\mathbf{r} = \int \mathbf{v}\, dt = (\frac{4}{3}t^3 - 1.5t^2 - 3t)\mathbf{i} + (0.5t^4 - 2t^2 + 4t)\mathbf{j}$

At t = 3: $\mathbf{r} = (13.5\mathbf{i} + 34.5\mathbf{j})$

Answer: (a) $(24\mathbf{i} + 46\mathbf{j})$ m s^{-1} (b) $(13.5\mathbf{i} + 34.5\mathbf{j})$ m

3 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The heights of plants, in cm are normally distributed. 35% are less than 47 and 35% are greater than 54.

Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 47) = 0.35$ gives $\frac{47 - \mu}{\sigma} = -0.3853$

$P(X > 54) = 0.35$ gives $\frac{54 - \mu}{\sigma} = 0.3853$

Subtract: $\frac{7}{\sigma} = 0.3853 - (-0.3853) = 0.7706$

$\sigma = 9.083$

$\mu = 47 - (-0.3853)(9.083) = 50.50$

Answer: $\mu = 50.5$, $\sigma = 9.08$

4 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{76 + 42\sqrt{3}}$ in the form

$a + b\sqrt{3}$, where a and b are positive integers.

Suppose $\sqrt{76 + 42\sqrt{3}} = a + b\sqrt{3}$. Square both sides.

$76 + 42\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}$.

Equate: $a^2 + 3b^2 = 76$ and $2ab = 42$, so $ab = 21$.

Try $a = 7$, $b = 3$: $49 + 27 = 76$ □.

$(a > b\sqrt{3}$, so the root is positive.)

Answer: $7 + 3\sqrt{3}$

5 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\tan\theta + \cot\theta \equiv 2\,\mathrm{cosec}\,2\theta$.

(b) Hence solve $\tan\theta + \cot\theta = 3$ for $0 \leq \theta < 360°$ (1 d.p.).

$\tan\theta + \cot\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\,\mathrm{cosec}\,2\theta$

$2\,\mathrm{cosec}\,2\theta = 3$, so $\sin 2\theta = \frac{2}{3}$, $2\theta = 41.8°, 138.2°, 401.8°, 498.2°$

Halve each value

Answer: $\theta = 20.9°, 69.1°, 200.9°, 249.1°$

6 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class P has 20 students with mean mark 77 and standard deviation 6. Class R has 30 students with mean mark 68 and standard deviation 7.
(a) Find the mean mark of all 50 students.
(b) Find the standard deviation of all 50 students.

Σx for P = $20 \times 77 = 1540$, Σx for R = $30 \times 68 = 2040$

Combined mean = $\frac{3580}{50} = 71.6$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $20(36 + 5929) + 30(49 + 4624) = 259490$

Variance = $\frac{259490}{50} - 71.600^2 = 63.240$, so $\sigma = 7.952$

Answer: (a) 71.6 (b) 7.95

7 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 3}{x - 3}$ for $x \neq 3$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 3) = kx - 3$, so $x(y - k) = 3y - 3$.

$f^{-1}(x) = \frac{3x - 3}{x - k}$.

$f = f^{-1}$ requires $k = 3$.

$f(x) = x$: $3x - 3 = x(x - 3)$, so $x^2 - 6x + 3 = 0$.

$x = 3 \pm \sqrt{6} = 3 \pm \sqrt{6}$.

Answer: $k = 3$; $x = 3 \pm \sqrt{6}$

8 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \sqrt{x}e^x$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (\sqrt{x}e^x)^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (\frac{1}{4} + \frac{3}{4}e^4)$

Answer: $\frac{1}{4}\pi(1 + 3e^4)$

9 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 110 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{5}{12}$ and $\mu = 0.2$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Normal reaction: $R = 110\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 110\sin\alpha$, so $H(\frac{12}{13} + 0.2 \times \frac{5}{13}) = 110(\frac{5}{13} - 0.2 \times \frac{12}{13})$

$H_{min} = 22$

About to slide up: $H\cos\alpha = 110\sin\alpha + \mu R$, so $H(\frac{12}{13} - 0.2 \times \frac{5}{13}) = 110(\frac{5}{13} + 0.2 \times \frac{12}{13})$

$H_{max} = 74$

Answer: $22 \leq H \leq 74$

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix}-4\\2\\-5\end{pmatrix} + t\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-8\\1\\-2\end{pmatrix} + t\begin{pmatrix}-1\\2\\-3\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-4\\-1\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\-2\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}1\\0\\-2\end{pmatrix} = 0$

$-10 + t(5) = 0$, so $t = 2$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-2\\-1\\-1\end{pmatrix}$, with length $\sqrt{6}$

Answer: t = 2, minimum distance = $\sqrt{6}$ m