Day 47 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A colony of 200 cells grows by 12% each day.
Find the least whole number of days for the colony to exceed 1000 cells.

$200 \times 1.12^n > 1000$, so $1.12^n > 5$

$n > \dfrac{\ln 5}{\ln 1.12} = 14.20$

n must be a whole number, so n = 15

Answer: 15

2 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 10 m s^{-1} from the top of a cliff. The top of the cliff is 14.1 m above the sea.
Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-14.1 = 10t - 4.9t^2$, so $4.9t^2 - 10t - 14.1 = 0$

$t = \dfrac{10 + \sqrt{(10^2 + 19.6 \times 14.1)}}{9.8} = \dfrac{10 + 19.4}{9.8} = 3$

Height risen above the cliff top = $\dfrac{10^2}{(2 \times 9.8)} = 5.10$ m

Total distance = $2 \times 5.10 + 14.1$

Answer: (a) 3 s (b) 24.3 m

3 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - 3x - 4 \geq 0$ and $x - 1 > -6$.

Quadratic: $(x + 1)(x - 4) \geq 0$ gives $x \leq -1$ or $x \geq 4$.
Linear: $x - 1 > -6$ gives $x > -5$.
Both must hold: take the overlap of the two sets.

Answer: $-5 < x \leq -1$ or $x \geq 4$

4 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 10 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 100$, and after 6 minutes $\theta = 70$.
Find A and k, then find when $\theta = 31$ (3 s.f.).

t = 0: $10 + A = 100$ so A = 90

t = 6: $10 + 90e^{-6k} = 70$ so $e^{-6k} = \dfrac{2}{3}$

$k = \dfrac{1}{6} \ln \dfrac{3}{2} = 0.0676$

$90\, e^{-kt} = 21$ so $t = \dfrac{1}{k} \ln \dfrac{30}{7}$

t = 21.5 minutes

Answer: A = 90, k = 0.0676, t = 21.5 minutes

5 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(3\mathbf{i})$ m with velocity $(10\mathbf{i} + 10\mathbf{j})$ m s^{-1}. At t = 8, P has velocity $(-14\mathbf{i} - 14\mathbf{j})$ m s^{-1}.
Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{i} + \mathbf{j})$, (c) the position vector of P at that time.

$\mathbf{a} = \dfrac{\mathbf{v} - \mathbf{u}}{t} = \dfrac{(-14\mathbf{i} - 14\mathbf{j}) - (10\mathbf{i} + 10\mathbf{j})}{8} = (-3\mathbf{i} - 3\mathbf{j})$

$\mathbf{v} = (10\mathbf{i} + 10\mathbf{j}) + t(-3\mathbf{i} - 3\mathbf{j})$

Parallel to $(\mathbf{i} + \mathbf{j})$: $(-3t + 10) \times 1 = (-3t + 10) \times 1$, giving t = 2

$\mathbf{r} = (3\mathbf{i}) + 2(10\mathbf{i} + 10\mathbf{j}) + \frac{1}{2} \times 2^2 (-3\mathbf{i} - 3\mathbf{j}) = (17\mathbf{i} + 14\mathbf{j})$

Answer: (a) $(-3\mathbf{i} - 3\mathbf{j})$ m s^{-2} (b) t = 2 (c) $(17\mathbf{i} + 14\mathbf{j})$ m

6 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is N(μ, 25) with known variance 25. The mean was claimed to be 37 ml.
A random sample of 36 has mean 34.75. Test at the 5% level whether the mean is different from 37.

H_0: $\mu = 37$, H_1: $\mu \neq 37$. Under H_0, the sample mean ~ $N(37, \frac{25}{36})$

$z = \frac{34.75 - 37}{5 \div \sqrt{36}} = -2.700$

One tail: $P(Z < -2.700) = 0.00347$, compared with 2.5%

$0.00347 < 2.5\%$, so reject H_0

There is sufficient evidence at the 5% level that the mean has changed as claimed

Answer: Reject H_0 ($z = -2.700$, p-value 0.00347)

7 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(3 + 5x)^{-1}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(3 + 5x)^{-1} = 3^{-1}(1 + \frac{5}{3}x)^{-1} = \frac{1}{3}(1 + u)^{-1}$, $u = \frac{5}{3}x$.

$(1 + u)^{-1} = 1 + (-1)u + 1u^2 + (-1)u^3 + ...$

Substitute u and multiply by $\frac{1}{3}$.

Valid when $|u| < 1$, i.e. $|x| < \frac{3}{5}$.

Answer: $\frac{1}{3} - \frac{5}{9}x + \frac{25}{27}x^2 - \frac{125}{81}x^3 + ...$; $|x| < \frac{3}{5}$

8 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(60, 0.25).
(a) State why a normal approximation is suitable for X.
(b) Use a normal approximation to estimate $P(X \geq 16)$, to 3 significant figures.

np = 15 and nq = 45, both greater than 5 (n large, p not extreme)

Y ~ N(15, 11.25) with $\sigma = 3.354$

Apply the continuity correction:

$P(X \geq 16) \approx P(Y > 15.5) = P(Z > 0.149) = 0.441$

Answer: 0.441

9 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 14x + 28}{(x + 1)(x + 4)} \equiv 1 + \frac{A}{x + 1} + \frac{B}{x + 4}$.

(b) Hence find the exact value of $\int_0^1 \frac{x^2 + 14x + 28}{(x + 1)(x + 4)}\,dx$.

Numerator = $(x + 1)(x + 4) + A(x + 4) + B(x + 1)$

$x = -1$ gives $A = 5$; $x = -4$ gives $B = 4$

Integral = $[x + 5\ln(x + 1) + 4\ln(x + 4)]$

Substitute limits 0 and 1

Answer: A = 5, B = 4; $1 - 3\ln 2 + 4\ln 5$

10 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = 3(t + \frac{1}{t})$, $y = 2(t - \frac{1}{t})$, $t \neq 0$.

Find a Cartesian equation of the curve.

$x^2 = 3^2(t^2 + 2 + \frac{1}{t^2})$.

$y^2 = 2^2(t^2 - 2 + \frac{1}{t^2})$.

$\frac{x^2}{9} - \frac{y^2}{4} = 4$.

Answer: $\mathbf{\frac{x^2}{9} - \frac{y^2}{4} = 4}$