Day 45 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^2 \frac{10x + 10}{x^2 + 2x + 2}\,dx$$

Numerator = 5 × derivative of denominator

Integral = $[5 \ln(x^2 + 2x + 2)]$

$= 5 (\ln 10 - \ln 2)$

Answer: $5\ln 5$

2 Poisson: scaling the rate, then binomial (CIE)

Customers arrive at a till at an average rate of 6 per hour, independently and at random.

(a) Find the probability of at least 2 in a 10-minute period.

(b) Find the probability that, of 9 separate 10-minute periods, exactly 3 have no arrivals.

10-minute period: $\lambda = 6 \times \frac{10}{60} = 1$, so $X \sim Po(1)$

(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.2642$

(b) $P(X = 0) = e^{-1} = 0.3679$

$Y \sim B(9, 0.3679)$: $P(Y = 3) = 0.267$

Answer: (a) 0.264 (b) 0.267

3 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 75 m from O, and reaches a maximum height of 25 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 25$, so $u_y = 22.14$

Time of flight $= \frac{2u_y}{9.8} = 4.518$ s, so $u_x = \frac{75}{4.518} = 16.60$

$\tan\theta = \frac{u_y}{u_x} = 1.333$, so $\theta = 53.1°$

Speed $= \sqrt{(u_x^2 + u_y^2)} = 27.7$

Answer: $\theta = 53.1°$, speed 27.7 m s^{-1}

4 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of

y = 2f(x + 2) + 3.

Inside f: replace x by x + 2, so x-coordinates become (x − 2) ÷ 1.

Outside: y-coordinates become 2 × y + 3.

A: (−8, 5) B: (−6, −3)

C: (−2, −1) D: (4, −5)

Answer: A′(−8, 5), B′(−6, −3), C′(−2, −1), D′(4, −5)

5 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{4}$ and $\tan B = \frac{3}{5}$.

Find tan(A + B) and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{3}{20}} = \frac{\frac{17}{20}}{\frac{17}{20}} = 1$$

A + B is acute, so $A + B = \pi/4$

Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$

6 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-2t^2 + 8t)\mathbf{i} + (-t^2 - 2t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + 2\mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-4t + 8)\mathbf{i} + (-2t - 2)\mathbf{j}$

Parallel to $(\mathbf{i} + 2\mathbf{j})$: $(-4t + 8) \times 2 = (-2t - 2) \times 1$

$t(-6) = -18$, so $t = 3$

At $t = 3$: $\mathbf{v} = (-4\mathbf{i} - 8\mathbf{j})$, speed $= \sqrt{(-4^2 + -8^2)}$

Answer: (a) t = 3 (b) 8.94 m s^{-1}

7 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(12, \frac{1}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{12 - r}{r + 1} \times \frac{1}{4}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(12, r + 1)}{C(12, r)} \times \frac{p}{q} = \frac{12 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{5}$ and $q = \frac{4}{5}$, $p \div q = \frac{1}{4}$

The ratio exceeds 1 when $(12 - r)\frac{1}{5} > (r + 1)\frac{4}{5}$, that is $r < 1.6$

So probabilities rise up to r = 2 and then fall; the mode is 2

$P(X = 2) = 0.283$

Answer: 2 with probability 0.283

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x + 3y = -2$

$x^2 + 2xy - y^2 = 14$

From the first equation $x = -2 - 3y$.

Substitute into the second equation and collect terms:

this gives $2y^2 + 8y - 10 = 0$.

$y = -5$ or $y = 1$, giving $x = 13$ or $x = -5$.

Answer: (x, y) = (13, −5) and (−5, 1)

9 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = (\theta - \sin\theta)$, $y = (1 - \cos\theta)$.

Find the exact area under one arch, for $0 \le \theta \le 2\pi$,

between the curve and the x-axis.

$dx/d\theta = (1 - \cos\theta)$

Area $= \int_0^{2\pi} 1^2(1 - \cos\theta)^2 \, d\theta = \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, so the integrand integrates to $2\pi + \frac{1}{2} \times 2\pi = 3\pi$

Answer: 3π

10 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $12y = 5x + k$ is a tangent to the circle
$(x + 3)^2 + (y + 4)^2 = 16$.
Find the two possible values of k.

Substitute $y = \frac{5x + k}{12}$: $(x + 3)^2 + (\frac{5x + k}{12} + 4)^2 = 16$.

Multiply by 144 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 33)^2 = 2704$.

$k + 33 = \pm 52$.

Answer: $k = -85$ or $k = 19$