Day 43 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^2 + 26x - 31}{(x + 3)(x - 2)^2}$ in the form $\frac{A}{(x + 3)} + \frac{B}{(x - 2)} + \frac{C}{(x - 2)^2}$.

$x^2 + 26x - 31 \equiv A(x - 2)^2 + B(x + 3)(x - 2) + C(x + 3)$.

Put $x = -3$: $-100 = A(-5)^2$, so $A = -4$.

Put $x = 2$: $25 = C(5)$, so $C = 5$.

Compare x^2: $1 = 1A + 1B$, so $B = 5$.

Answer: A = −4, B = 5, C = 5

2 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (2, 12) and (5, 75).
Find the values of a and n.

$12 = a \times 2^n$ and $75 = a \times 5^n$

Divide: $\left(\frac{5}{2}\right)^n = \frac{25}{4}$ so $n = 2$

$a = 12 \div 2^2 = 3$

Answer: a = 3, n = 2

3 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(2, 1) lies on the circle: $(2 - r)^2 + (1 - r)^2 = r^2$.

$r^2 - 6r + 5 = 0$, so $r = 1$ or $r = 5$.

The diagram shows P on the smaller circle, so $r = 1$.

$(x - 1)^2 + (y - 1)^2 = 1$, then expand.

Answer: $x^2 + y^2 - 2x - 2y + 1 = 0$

4 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{1}{33 + 21 \sin\theta + 20 \cos\theta}$

(a) Write $21 \sin\theta + 20 \cos\theta$ as $R \sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{21^2 + 20^2} = 29$, $\tan\alpha = \frac{20}{21}$, $\alpha = 43.6°$

Denominator $= 33 + 29 \sin(\theta + 43.6°)$, which lies between 4 and 62

Greatest $f(\theta) = \frac{1}{4} = \frac{1}{4}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{1}{62} = \frac{1}{62}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 43.6° = 270°$, so $\theta = 226.4°$

Answer: (a) $29 \sin(\theta + 43.6°)$ (b) greatest $\frac{1}{4}$, least $\frac{1}{62}$, $\theta = 226.4°$

5 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(32, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 56)$.

The area 0.7 leaves 0.300 in total in the tails, 0.150 in each

$P(X < 17) = 0.150$ so $\frac{17 - 32}{\sigma} = -1.0364$

$\sigma = \frac{15}{1.0364} = 14.47$

$P(X > 56) = P(Z > 1.658) = 0.0486$

Answer: (a) 14.5 (b) 0.0486

6 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-2t^2 + 9t)\mathbf{i} + (-0.5t^2 - 4t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} - \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-4t + 9)\mathbf{i} + (-t - 4)\mathbf{j}$

Parallel to $(\mathbf{i} - \mathbf{j})$: $(-4t + 9) \times -1 = (-t - 4) \times 1$
$t(5) = 5$, so $t = 1$

At $t = 1$: $\mathbf{v} = (5\mathbf{i} - 5\mathbf{j})$, speed $= \sqrt{(5^2 + -5^2)}$

Answer: (a) $t = 1$ (b) 7.07 m s^{-1}

7 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 10 times in round 1 and 7 times in round 2.
Given that exactly 7 wins occur altogether, find the probability that exactly 3 of them were in round 1.

Let $X \sim B(10, p)$, $Y \sim B(7, p)$ be independent; $X + Y \sim B(17, p)$

$P(X = 3 \mid X + Y = 7) = \frac{P(X = 3)P(Y = 4)}{P(X + Y = 7)}$

$= \frac{C(10,3)C(7,4)\, p^7(1 - p)^{10}}{C(17,7)\, p^7(1 - p)^{10}}$

The powers of p and $(1 - p)$ cancel, leaving $\frac{120 \times 35}{19448}$

Answer: $\frac{525}{2431} = 0.216$

8 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 4x^2 - 16x + 20$
for $x \leq k$. Given that k is the greatest value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 4(x - 2)^2 + 4$.
The vertex is at $x = 2$, so $k = 2$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 4 = 4(x - 2)^2$, so $(x - 2) = \pm\sqrt{\frac{y - 4}{4}}$.

As $x \leq 2$, take the negative root: $x = 2 - \sqrt{\frac{y - 4}{4}}$.

Range of f = domain of f^{-1}: $f(2) = 4$, so $x \geq 4$.

Answer: $k = 2$; $f^{-1}(x) = 2 - \sqrt{\frac{x - 4}{4}}$; $x \geq 4$

9 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (5 kg) and B (5 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 3 m above the ground.

Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 5 \times 9.8 \times \frac{1}{2} = 5a$. B: $5 \times 9.8 - T = 5a$

Adding: $9.8(5 - 5 \times \frac{1}{2}) = 10a$, so $a = 2.45$

$T = 5(9.8 - 2.45) = 36.75$

When B lands: $v^2 = 2 \times 2.45 \times 3 = 14.7$

Then A decelerates at $g \sin \alpha = 4.9$: further distance $= \frac{14.7}{2 \times 4.9} = 1.5$

Answer: 1.5 m

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix}6\\4\\4\end{pmatrix} + t\begin{pmatrix}3\\3\\0\end{pmatrix}$.

The point P has coordinates (2, 2, 0).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix}-4\\-2\\-4\end{pmatrix} - t\begin{pmatrix}3\\3\\0\end{pmatrix}$

$\mathrm{FP} \cdot \mathbf{d} = 0$: $-18 - t(18) = 0$, so $t = -1$

F = (3, 1, 4)

$\mathrm{PF} = \begin{pmatrix}-1\\1\\-4\end{pmatrix}$, so $\mathrm{PF} = \sqrt{18}$

Answer: F = (3, 1, 4), PF = $3\sqrt{2}$