Day 42 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = 2t^2$. The line $y = 2x + 4$
meets C at A and B. Find the exact length AB.

Substitute: $2t^2 = 2t + 4$.
$2t^2 - 2t - 4 = 0 \Rightarrow t = 2$ or $t = -1$.
A(2, 8), B(−1, 2).
$AB^2 = 3^2 + 6^2 = 45$.
Answer: $\mathbf{AB = 3\sqrt{5}}$

2 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 10 cm and its area is 6 cm^2.
Find the two possible pairs of values of r and θ.

Arc = rθ, so $2r + r\theta = 10$ and $\frac{1}{2}r^2\theta = 6$

$r\theta = 10 - 2r$, so $\frac{1}{2}r(10 - 2r) = 6$

$r^2 - 5r + 6 = 0$, so $(r - 2)(r - 3) = 0$

r = 2: rθ = 6, θ = 3 r = 3: rθ = 4, $\theta = \frac{4}{3}$

Answer: $\mathbf{r = 2, \theta = 3}$ **or** $\mathbf{r = 3, \theta = \frac{4}{3}}$

3 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school says that 25% of its pupils walk to school. 30 pupils are surveyed, and 13 walk.
Test, at the 5% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.25, H_1: p > 0.25. Let X ~ B(30, 0.25)
$P(X \geq 13) = 1 - P(X \leq 12) = 0.0216$
0.0216 is less than 5%, so reject H_0
There is sufficient evidence that the proportion is higher than the claimed 0.25
Answer: Reject H_0: p-value 0.0216 < 5%

4 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = e^y + 2y$.
Find dy/dx and d^2y/dx^2 when y = 0.

$dx/dy = e^y + 2 = 3$ when y = 0, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{3}$

$d^2x/dy^2 = 1$ when y = 0

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{1}{3^3}$

Answer: $\mathbf{dy/dx = \frac{1}{3}}$, $\mathbf{d^2y/dx^2 = -\frac{1}{27}}$

5 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (2 kg) lies on a rough table with $\mu = 0.4$, and is joined by a light string over a smooth pulley (see diagram) to particle B (5 kg) on a smooth slope, $\sin\alpha = \frac{3}{5}$.

Released from rest, the string is cut when B has moved 3.5 m. Find the further distance A slides before stopping.

A: $T - 0.4 \times 2 \times 9.8 = 2a$. B: $5 \times 9.8 \times \frac{3}{5} - T = 5a$

Adding: $9.8(5 \times \frac{3}{5} - 0.4 \times 2) = 7a$, so $a = 3.08$

$T = 2(3.08 + 0.4 \times 9.8) = 14$

When cut: $v^2 = 2 \times 3.08 \times 3.5 = 21.56$. Then A decelerates at $\mu g = 3.92$

Further distance $= \frac{21.56}{2 \times 3.92} = 2.75$

Answer: 2.75 m

6 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $4k = n^2 + 1$ has no integer solutions (k and n integers).

Every integer is $4q$, $4q + 1$, $4q + 2$, $4q + 3$ for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 4$, so:

$0 \to 0$; $1 \to 1$; $2 \to 0$; $3 \to 1$ ($r \to n^2 \bmod 4$)

So $n^2 + 1 \equiv 1$ or $2 \pmod 4$.

No residue is 0, so $n^2 + 1$ is never a multiple of 4.

Answer: Proved

7 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At $t = 0$ seconds P is at $(3\mathbf{i})$ m with velocity $(5\mathbf{i} + 4\mathbf{j})$ m s^{-1}. At $t = 10$, P has velocity $(-5\mathbf{i} - 16\mathbf{j})$ m s^{-1}.

Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{i})$, (c) the position vector of P at that time.

$a = \frac{v - u}{t} = \frac{(-5\mathbf{i} - 16\mathbf{j}) - (5\mathbf{i} + 4\mathbf{j})}{10} = (-\mathbf{i} - 2\mathbf{j})$

$v = (5\mathbf{i} + 4\mathbf{j}) + t(-\mathbf{i} - 2\mathbf{j})$

Parallel to $(\mathbf{i})$: $(-t + 5) \times 0 = (-2t + 4) \times 1$, giving $t = 2$

$r = (3\mathbf{i}) + 2(5\mathbf{i} + 4\mathbf{j}) + \frac{1}{2} \times 2^2 (-\mathbf{i} - 2\mathbf{j}) = (11\mathbf{i} + 4\mathbf{j})$

Answer: (a) $(-\mathbf{i} - 2\mathbf{j})$ m s^{-2} (b) $t = 2$ (c) $(11\mathbf{i} + 4\mathbf{j})$ m

8 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{x^2 + 7}{x + 3}$ at the point where $x = 3$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 3$: $y = \frac{8}{3}$. Differentiate: gradient $= \frac{5}{9}$

Tangent: $y - \frac{8}{3} = \frac{5}{9}(x - 3)$, so $y = \frac{5}{9}x + 1$

y-intercept 1; x-intercept $-\frac{9}{5}$

Area $= \frac{1}{2} \times |1| \times |-\frac{9}{5}|$

Answer: $\frac{9}{10}$

9 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = t^2$, $y = 4t^3$.

Show that $y^2 = kx^3$ and find k.

$y^2 = 16t^6$.

$x^3 = t^6$.

So $y^2 = 16\,x^3$.

Answer: k = 16

10 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 6$ and $v = \frac{y}{2}$. The regression line of v on u is $v = 8 + 2u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when x = 10.

Substitute $u = x - 6$ and $v = y/2$: $\frac{y}{2} = 8 + 2(x - 6)$

$y = 2[8 + 2x - 12] = 2 \times -4 + 2 \times 2x$

$y = -8 + 4x$

At x = 10: $y = -8 + 4 \times 10 = 32$

Answer: (a) $y = -8 + 4x$ (b) 32