Day 40 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

15 recorded masses (grams) have mean 45 and standard deviation 4.
One value was recorded as 34 but should have been 26.
Find the correct mean and the correct standard deviation.

$\Sigma x = 15 \times 45 = 675$, so corrected $\Sigma x = 675 - 34 + 26 = 667$

Correct mean $= \frac{667}{15} = 44.47$

$\Sigma x^2 = 15(16 + 2025) = 30615$, corrected $\Sigma x^2 = 30615 - 1156 + 676 = 30135$

Variance $= \frac{30135}{15} - 44.467^2 = 31.716$, $\sigma = 5.632$

Answer: mean 44.5, standard deviation 5.63

2 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_1^e \frac{1}{x(\ln x + 2)}\,dx$$

Let $u = \ln x + 2$, $du = \frac{1}{x}dx$

Integral $= \int_2^3 \frac{1}{u}\,du = [\ln u]$

$= \ln 3 - \ln 2$

Answer: $-\ln 2 + \ln 3$

3 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{6}{1+3} = \frac{3}{2}$.

Gradient of $l_2 = -\frac{2}{3}$. Equation: $y - 6 = -\frac{2}{3}(x - 1)$.

$y = 0$: $x = 10$, so C(10, 0).

AB $= 2\sqrt{13}$, BC $= 3\sqrt{13}$: area = ½ × AB × BC.

Answer: C(10, 0); area 39

4 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\tan\theta + \cot\theta \equiv 2\,\text{cosec}\,2\theta$.
(b) Hence solve $\tan\theta + \cot\theta = 8$ for $0 \le \theta < 360°$ (1 d.p.).

$\tan\theta + \cot\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\,\text{cosec}\,2\theta$

$2\,\text{cosec}\,2\theta = 8$, so $\sin 2\theta = \frac{2}{8}$, $2\theta = 14.5°, 165.5°, 374.5°, 525.5°$

Halve each value

Answer: θ = 7.2°, 82.8°, 187.2°, 262.8°

5 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 + 1$. Also $fg(x) = -2x^2 + 2$, where f is
a linear function. (a) Find f(x).
(b) Solve $gf(x) = 5$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 + 1) + b = -2x^2 + 2$.
So $a = -2$ and $-2 + b = 2$, giving $b = 4$.
$f(x) = -2x + 4$.
(b) $gf(x) = (-2x + 4)^2 + 1 = 5$.
$(-2x + 4)^2 = 4$, so $-2x + 4 = \pm 2$.
$x = 1$ or $x = 3$.

Answer: (a) f(x) = −2x + 4 (b) x = 1 or x = 3

6 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 1 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 2.5 m s^{-2} to rest.
The journey takes 55 s and covers 820 m. Find V and the time spent at constant speed.

Time accelerating = V, time decelerating = $\frac{V}{2.5}$, so time at constant speed = $55 - V - \frac{V}{2.5}$

Area under the speed-time graph: $820 = 55V - \frac{V^2}{2} - \frac{V^2}{5}$

Multiply by 2ab: $3.5V^2 - 275V + 4100 = 0$

$V = 20$ or $V = \frac{410}{7}$

$V = \frac{410}{7}$ gives time at constant speed -27 s, which is negative, so $V = 20$

Time at constant speed = $55 - 20 - 8 = 27$ s

Answer: V = 20, 27 s

7 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−7, 12) and B(3, 12). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-7))^2 + 144 = (c - 3)^2 + 144$.
Expand and solve: $20c = -40$, so $c = -2$.
Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$
$= \frac{1}{2}|(-5) \times 12 - 5 \times 12| = 60$.

Answer: C(−2, 0); area 60

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{1}{34 + 20 \sin\theta + 21 \cos\theta}$

(a) Write $20 \sin\theta + 21 \cos\theta$ as $R \sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{20^2 + 21^2} = 29$, $\tan\alpha = \frac{21}{20}$, $\alpha = 46.4°$

Denominator = $34 + 29 \sin(\theta + 46.4°)$, which lies between 5 and 63

Greatest $f(\theta) = \frac{1}{5} = \frac{1}{5}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{1}{63} = \frac{1}{63}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 46.4° = 270°$, so $\theta = 223.6°$

Answer: (a) $29 \sin(\theta + 46.4°)$ (b) greatest $\frac{1}{5}$, least $\frac{1}{63}$, $\theta = 223.6°$

9 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is $f(x) = a + bx$ for $0 \le x \le 3$, and 0 otherwise, where a and b are constants. $E(X) = \frac{33}{20}$.

(a) Find a and b.
(b) Find $P(X < 1.5)$.

Total area = 1: $3a + 4.5b = 1$

$E(X) = \int x(a + bx)\,dx = 4.5a + 9b = \frac{33}{20}$

Solving the simultaneous equations: $a = \frac{7}{30}$, $b = \frac{1}{15}$

$P(X < 1.5) = \int f(x)\,dx$ from 0 to 1.5 $= \frac{7}{30} \times 1.5 + \frac{1}{15} \times 1.125$

$= \frac{17}{40}$

Answer: (a) $a = \frac{7}{30}$, $b = \frac{1}{15}$ (b) $\frac{17}{40}$

10 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 14.7^2 - 2 \times 9.8 \times H$, so H = 11.025

At height $\frac{H}{2}$ = 5.5125: $v_y^2 = 14.7^2 - 2 \times 9.8 \times 5.5125 = 108.045$

v_x = 19.6 (constant), so speed^2 = 384.16 + 108.045 = 492.205

Answer: (a) 11.025 m (b) 22.2 m s^{-1}