Day 4 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 red, 4 blue and 3 green counters. A player takes two without replacement. Two reds win £4, two blues win £7, anything else wins nothing.
(a) Find the expected winnings.
(b) The player pays £1 to play. Find the expected profit.

P(RR) = $\frac{3}{10} \times \frac{2}{9} = \frac{1}{15}$

P(BB) = $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$

(a) E = $4 \times \frac{1}{15} + 7 \times \frac{2}{15} = \frac{6}{5}$

(b) Expected profit = $\frac{6}{5} - 1 = \frac{1}{5}$

Answer: (a) £1.20 (b) £0.200

2 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 450 kg, carrying a passenger of mass 85 kg, is supported by a vertical cable. The lift is moving downwards and accelerating at 0.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is downwards
Passenger: $85 \times 9.8 - R = 85 \times 0.4$
$R = 85 \times (9.8 - 0.4) = 799$
Lift and passenger: $535 \times 9.8 - T = (450 + 85) \times 0.4$
$T = 535 \times 9.4 = 5029$
Answer: (a) 799 N (b) 5029 N

3 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that the difference between the squares of any two odd integers is a multiple of 8.

Let the integers be 2m + 1 and 2n + 1.
$(2m + 1)^2 - (2n + 1)^2 = 4m^2 + 4m - 4n^2 - 4n$
$= 4[m(m + 1) - n(n + 1)]$.
m(m + 1) is a product of consecutive integers so is even; same for n(n + 1).
So the bracket is even, and the expression is a multiple of 8.
Answer: Proved

4 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{29 + 7\sin\theta + 24\cos\theta}$
(a) Write $7\sin\theta + 24\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{7^2 + 24^2} = 25$, $\tan\alpha = \frac{24}{7}$, $\alpha = 73.7°$

Denominator = $29 + 25\sin(\theta + 73.7°)$, which lies between 4 and 54

Greatest $f(\theta) = \frac{4}{4} = 1$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{54} = \frac{2}{27}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 73.7° = 270°$, so $\theta = 196.3°$

Answer: (a) $25\sin(\theta + 73.7°)$ (b) greatest 1, least $\frac{2}{27}$, $\theta = 196.3°$

5 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 20 pairs of values: $\Sigma x = 180$, $\Sigma y = 1040$, $\Sigma x^2 = 1805$, $\Sigma xy = 9212$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 5 and 14. Estimate y when x = 15 and comment on its reliability.

$S_{xx} = 1805 - \frac{180^2}{20} = 185$

$S_{xy} = 9212 - \frac{180 \times 1040}{20} = -148$

$b = \frac{S_{xy}}{S_{xx}} = \frac{-148}{185} = -\frac{4}{5}$

$a = \bar{y} - b\,\bar{x} = 52 - (-\frac{4}{5})(9) = 59.2$

$y = 59.2 - 0.8x$
At x = 15: y = 47.2. This is extrapolation (outside the data range), so may be unreliable
Answer: (a) $y = 59.2 - 0.8x$ (b) 47.2, unreliable (extrapolation)

6 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $24 \sin\theta - 7 \cos\theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $24 \sin\theta - 7 \cos\theta = -11$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{24^2 + 7^2} = 25$ and $\tan\alpha = \frac{7}{24}$, so $\alpha = 16.3°$
So $24 \sin\theta - 7 \cos\theta \equiv 25 \sin(\theta - 16.3°)$
$\sin(\theta - \alpha) = \frac{-11}{25}$
$\theta - \alpha = -26.1°$ or $206.1°$ (using $\alpha = 16.3°$)
Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$
Answer: (a) R = 25, α = 16.3° (b) θ = 222.4°, 350.2°

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=5}^{11}(2 \times 2^r + 2r)$.

Geometric part: first term $2 \times 2^5 = 64$, ratio 2, 7 terms.
Sum $= \frac{64(2^7 - 1)}{1} = 8128$.
Arithmetic part: 2 × 7 terms, average (5 + 11)/2: sum = 112.
Total = 8128 + 112.
Answer: 8240

8 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x − 1)
the remainder is −4; when divided by (x − 2)
the remainder is −1. Find the remainder when
f(x) is divided by (x − 1)(x − 2).

$f(x) = (x - 1)(x - 2)Q(x) + px + q$.
$f(1) = -4$: $p + q = -4$.
$f(2) = -1$: $2p + q = -1$.
Subtracting: $-p = -3$, so $p = 3$, $q = -7$.
Answer: $3x - 7$

9 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x\sqrt{75 - x}$, for $0 < x < 75$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{2(150 - 3x)}{2\sqrt{75 - x}}$

$\frac{dy}{dx} = 0$: $150 - 3x = 0$, so $x = 50$

$y = 2 \times 50 \times 5$

Gradient changes from positive to negative: a maximum

Answer: (50, 500), maximum

10 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (4 kg) and B (8 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\sin\alpha = \frac{4}{5}$, and B is 1 m above the ground.

Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 4 \times 9.8 \times \frac{4}{5} = 4a$. B: $8 \times 9.8 - T = 8a$

Adding: $9.8(8 - 4 \times \frac{4}{5}) = 12a$, so $a = 3.92$

$T = 8(9.8 - 3.92) = 47.04$

When B lands: $v^2 = 2 \times 3.92 \times 1 = 7.84$

Then A decelerates at $g \sin\alpha = 7.84$: further distance $= \frac{7.84}{2 \times 7.84} = 0.5$

Answer: 0.5 m