Day 38 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 6π cm^{3}s^{-1}. Find the rate at which the depth h is rising when h = 4 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $6\pi = \frac{\pi}{4} \times 16 \times dh/dt$

$dh/dt = \frac{24}{16}$

Answer: $\frac{3}{2}$ cm s^{-1}

2 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ab^x$. The graph of lg y against x is the straight line shown.
Find a and b to 3 significant figures.

lg y = lg a + x lg b. The intercept is lg a = 0.7, so $a = 10^{0.7} = 5.012$

Gradient = $\frac{2.3 - 0.7}{8} = 0.2$

lg b = 0.2 so $b = 10^{0.2} = 1.585$

Answer: a = 5.01, b = 1.58

3 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -1$ and $u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find $\sum_{r=1}^{101} u_r$.

$u_1 = -1$, $u_2 = \frac{1}{2}$, $u_3 = 2$, $u_4 = -1$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $\frac{3}{2}$.

101 = 3 × 33 + 2: 33 full periods plus the first 2 terms.

Answer: 49

4 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 2.2 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 2 m s^{-2} to rest.
The journey takes 71 s and covers 1331 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{2.2}$, time decelerating = $\frac{V}{2}$, so time at constant speed = $71 - \frac{V}{2.2} - \frac{V}{2}$

Area under the speed-time graph: $1331 = 71V - \frac{V^2}{4.4} - \frac{V^2}{4}$

Multiply by 2ab: $4.2V^2 - 624.8V + 11712.8 = 0$

$V = 22$ or $V = \frac{2662}{21}$

$V = \frac{2662}{21}$ gives time at constant speed −50 s, which is negative, so V = 22

Time at constant speed = 71 − 10 − 11 = 50 s

Answer: V = 22, 50 s

5 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 3$ to find the exact value of

$\int_0^1 x(x^2 + 3)^2 \, dx$

$du = 2x \, dx$; limits $x = 0, 1$ become $u = 3, 4$

Integral $= \frac{1}{2}\int_3^4 u^2 \, du = [\frac{u^3}{6}]$

Evaluate the limits

Answer: $\frac{37}{6}$

6 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{42 - 12\sqrt{6}}$ in the form

$a - b\sqrt{6}$, where a and b are positive integers.

Suppose $\sqrt{42 - 12\sqrt{6}} = a - b\sqrt{6}$. Square both sides.

$42 - 12\sqrt{6} = a^2 + 6b^2 - 2ab\sqrt{6}$.

Equate: $a^2 + 6b^2 = 42$ and $2ab = 12$, so $ab = 6$.

Try $a = 6, b = 1$: $36 + 6 = 42$ ☐.

($a > b\sqrt{6}$, so the root is positive.)

Answer: $6 - 1\sqrt{6}$

7 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 48 m from O, and reaches a maximum height of 9 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 9$, so $u_y = 13.28$

Time of flight $= \frac{2u_y}{9.8} = 2.711$ s, so $u_x = \frac{48}{2.711} = 17.71$

$\tan\theta = \frac{u_y}{u_x} = 0.7500$, so $\theta = 36.9°$

Speed $= \sqrt{(u_x^2 + u_y^2)} = 22.1$

Answer: $\theta = 36.9°$, **speed 22.1 m s^{-1}**

8 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^3 + 3x^2 - 10x - 24 > 0$.

Use the factor theorem to find roots: $-4, -2, 3$.

$f(x) = (x + 4)(x + 2)(x - 3)$.

Sign of f in each region (left to right): $-, +, -, +$.

Take the regions where $f > 0$ (include roots if $\leq$ or $\geq$).

Answer: $-4 < x < -2$ **or** $x > 3$

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{-3x} \cos 2x$. Show that $dy/dx = e^{-3x}(-3\cos 2x - 2\sin 2x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (-3)e^{-3x}\cos 2x - e^{-3x} \times 2\sin 2x$

$dy/dx = 0$ requires $-3\cos 2x = 2\sin 2x$

$\tan 2x = -\frac{3}{2}$, smallest positive $2x = 2.1588$

$x = 1.08$

Second derivative test (or gradient change): a minimum

Answer: x = 1.08, minimum

10 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(80, 0.4)$.

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate $P(24 \leq X \leq 36)$, to 3 significant figures.

$np = 32$ and $nq = 48$, both greater than 5 (n large, p not extreme)

$Y \sim N(32, 19.2)$ with $\sigma = 4.382$

Apply the continuity correction:

$P(24 \leq X \leq 36) \approx P(23.5 < Y < 36.5) = 0.8478 - 0.02620 = 0.822$

Answer: 0.822