Day 37 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=4}^{10}(4 \times 3^r + 2r)$.

Geometric part: first term $4 \times 3^4 = 324$, ratio 3, 7 terms.

Sum = $\frac{324(3^7 - 1)}{2} = 354132$.

Arithmetic part: 2 × 7 terms, average (4 + 10)/2: sum = 98.

Total = 354132 + 98.

Answer: 354230

2 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A manufacturer says that 40% of its screens have no flaws. A sample of 25 screens is checked, and 16 have no flaws. Test, at the 5% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.4, H_1: p > 0.4. Let X ~ B(25, 0.4)

$P(X \geq 16) = 1 - P(X \leq 15) = 0.0132$

0.0132 is less than 5%, so reject H_0

There is sufficient evidence that the proportion is higher than the claimed 0.4

Answer: Reject H_0: p-value 0.0132 < 5%

3 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 3) + \log_2(x - 4) = 3$,
justifying clearly any root that you reject.

$(x + 3)(x - 4) = 2^3 = 8$

$x^2 - 1x - 20 = 0$

$(x - 5)(x + 4) = 0$, so x = 5 or x = −4

x = −4 makes x + 3 = −1 < 0 and log of a negative number is undefined

Answer: x = 5

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 33 and upper quartile 51. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 8 observations have mean 38. The largest is 88, an outlier, and it is removed. Find the new mean.

IQR = 51 − 33 = 18, so 1.5 × IQR = 27

Lower limit = 33 − 27 = 6, upper limit = 51 + 27 = 78

Σx = 8 × 38 = 304, so without 88: Σx = 216

New mean = $\frac{216}{7}$

Answer: (a) 6 to 78 (b) 30.86

5 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 7 m s^{-1} from the top of a cliff. The top of the cliff is 23.1 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-23.1 = 7t - 4.9t^2$, so $4.9t^2 - 7t - 23.1 = 0$

$t = \frac{7 + \sqrt{(7^2 + 19.6 \times 23.1)}}{9.8} = \frac{7 + 22.4}{9.8} = 3$

Height risen above the cliff top = $\frac{7^2}{(2 \times 9.8)} = 2.50$ m

Total distance = 2 × 2.50 + 23.1

Answer: (a) 3 s (b) 28.1 m

6 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\cos A = \frac{15}{17}$ and $\cos B = \frac{24}{25}$.

Find the exact values of $\cos(A - B)$ and $\tan(A - B)$.

$\cos A = \frac{15}{17}$, $\sin A = \frac{8}{17}$; $\sin B = \frac{7}{25}$, $\cos B = \frac{24}{25}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{416}{425}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\frac{87}{425}}{\frac{416}{425}} = \frac{87}{416}$

Answer: $\cos(A - B) = \frac{416}{425}$, $\tan(A - B) = \frac{87}{416}$

7 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{3}{4}u_n + \frac{3}{2}$, $u_1 = 26$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.01$.

(a) At the limit $L = \frac{3}{4}L + \frac{3}{2}$, so $L(1 - \frac{3}{4}) = \frac{3}{2}$.

$L = 6$.

(b) $u_n - L = (u_1 - L)(\frac{3}{4})^{n-1} = 20(\frac{3}{4})^{n-1}$.

Need $20 \times (\frac{3}{4})^{n-1} < 0.01$, so $n - 1 > 26.421$.

The least n is 28.

Answer: (a) L = 6 (b) n = 28

8 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 85 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.5$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 85\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 85\sin\alpha$, so $H(\frac{3}{5} + 0.5 \times \frac{4}{5}) = 85(\frac{4}{5} - 0.5 \times \frac{3}{5})$

$H_{min} = 42.5$

About to slide up: $H\cos\alpha = 85\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.5 \times \frac{4}{5}) = 85(\frac{4}{5} + 0.5 \times \frac{3}{5})$

$H_{max} = 467.5$

Answer: $42.5 \leq H \leq 467.5$

9 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(2, 4) lies on the circle: $(2 - r)^2 + (4 - r)^2 = r^2$.

$r^2 - 12r + 20 = 0$, so $r = 2$ or $r = 10$.

The diagram shows P on the smaller circle, so $r = 2$.

$(x - 2)^2 + (y - 2)^2 = 4$, then expand.

Answer: $x^2 + y^2 - 4x - 4y + 4 = 0$

10 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Squares of side x cm are cut from the corners of a square sheet of side 24 cm and the sides folded up to make an open box.
Find the value of x that maximises the volume, and that maximum volume.

$V = x(24 - 2x)^2 = 4x^3 - 96x^2 + 576x$

$dV/dx = 12x^2 - 192x + 576 = (24 - 2x)(24 - 6x) = 0$

$x < 12$, so $x = 4$

$d^2V/dx^2 = 24x - 192$, negative at $x = 4$: maximum

$V = 4 \times 16^2$

Answer: $x = 4$, $V = 1024\ cm^3$