Day 364 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = \sqrt{-x^2 + x + 30}$.
Find the largest possible domain of g, and the corresponding range.

Need $-x^2 + x + 30 \geq 0$, i.e. $(x + 5)(x - 6) \leq 0$.
Domain: $-5 \leq x \leq 6$.

Complete the square: $-x^2 + x + 30 = \frac{121}{4} - (x - \frac{1}{2})^2$.

Greatest value of $g = \sqrt{\frac{121}{4}} = \frac{11}{2}$, and least value is 0.

Answer: $-5 \leq x \leq 6$; $0 \leq g(x) \leq \frac{11}{2}$

2 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $15\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $15\cos\theta + 8\sin\theta = 16$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{15^2 + 8^2} = 17$ and $\tan\alpha = \frac{8}{15}$, so $\alpha = 28.1°$

So $15\cos\theta + 8\sin\theta \equiv 17\cos(\theta - 28.1°)$

$\cos(\theta - \alpha) = \frac{16}{17}$

$\theta - \alpha = \pm 19.7°$ (using $\alpha = 28.1°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 17, α = 28.1° (b) θ = 8.3°, 47.8°

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A javelin is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{5}{12}$. Find (a) the time of flight, (b) the range, (c) the greatest height. Take g = 9.8 m s^{-2}.

Components: horizontal 58.8, vertical 24.5 (as $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$)

Time of flight: $0 = 24.5\,t - 4.9t^2$, so $t = \frac{24.5}{4.9} = 5$

Range = $58.8 \times 5 = 294$

Greatest height: $0 = 24.5^2 - 2 \times 9.8 \times H$, so $H = \frac{600.25}{19.6} = 30.625$

Answer: (a) 5 s (b) 294 m (c) 30.625 m

4 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is N(μ, 16) with known variance 16. The mean was claimed to be 44 ml.
A random sample of 9 has mean 40.25. Test at the 5% level whether the mean is greater than 44.

H_0: $\mu = 44$, H_1: $\mu > 44$. Under H_0, the sample mean ~ N(44, $\frac{16}{9}$)

$z = \frac{40.25 - 44}{4 \div \sqrt{9}} = -2.812$

p-value = $P(Z > -2.812) = 0.998$
$0.998 > 5\%$, so do not reject H_0
There is insufficient evidence at the 5% level that the mean has changed as claimed

Answer: Do not reject H_0 ($z = -2.812$, p-value 0.998)

5 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\sec^2\theta - 5\tan\theta - 5 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $2\tan^2\theta - 5\tan\theta - 3 = 0$
$(2\tan\theta + 1)(\tan\theta - 3) = 0$

$\tan\theta = -\frac{1}{2}$ or $\tan\theta = 3$

$\theta = 153.4°$ and $71.6°$ $(+180°)$
Answer: θ = 71.6°, 153.4°, 251.6°, 333.4°

6 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - 2y = 8$
$x^2 - 2xy + y^2 = 16$

From the first equation $x = 8 + 2y$.
Substitute into the second equation and collect terms:
this gives $y^2 + 16y + 48 = 0$.
$y = -12$ or $y = -4$, giving $x = -16$ or $x = 0$.
Answer: (x, y) = (−16, −12) and (0, −4)

7 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(12\mathbf{i} + 29\mathbf{j})$ m s^{-1} and $(16\mathbf{i} + 32\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 5 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $12t\mathbf{i} + (29t - 4.9t^2)\mathbf{j}$; position of B: $16t\mathbf{i} + (32t - 4.9t^2)\mathbf{j}$
Subtracting, the $4.9t^2$ terms cancel: AB = $t(-4\mathbf{i} - 3\mathbf{j})$
After 5 s: AB = $(-20\mathbf{i} - 15\mathbf{j})$
Distance = $\sqrt{(20^2 + 15^2)} = 25$

The direction of AB is that of $(-4\mathbf{i} - 3\mathbf{j})$, which does not depend on t: tan angle = $\frac{3}{4}$

Answer: (a) 25 m (b) 36.9°

8 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{4}u_n + 12$, $u_1 = 25$.

(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.01$.

(a) At the limit $L = \frac{1}{4}L + 12$, so $L(1 - \frac{1}{4}) = 12$.

$L = 16$.

(b) $u_n - L = (u_1 - L)(\frac{1}{4})^{n-1} = 9(\frac{1}{4})^{n-1}$.

Need $9 \times (\frac{1}{4})^{n-1} < 0.01$, so $n - 1 > 4.907$.

The least n is 6.
Answer: (a) L = 16 (b) n = 6

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9 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(4) and Y ~ Po(1).
(a) State the distribution of X + Y.
(b) Given that X + Y = 3, find the probability that X = 2.

X + Y ~ Po(5)

$P(X = 2 \mid X + Y = 3) = \frac{P(X = 2)P(Y = 1)}{P(X + Y = 3)}$

$= \frac{e^{-4}4^2}{2! \times e^{-1}1^1/1!} \div \frac{e^{-5}5^3}{3!}$

$= C(3, 2)\ (\frac{4}{5})^2\ (\frac{1}{5})^1$, a binomial probability

= 0.384

Answer: (a) Po(5) (b) 0.384

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $\frac{x - 1}{(x + 5)(x + 3)}$ in partial fractions.

(b) Hence find the exact value of $\int_0^2 \frac{x - 1}{(x + 5)(x + 3)}\ dx$.

Let N/D = $\frac{A}{x + 5} + \frac{B}{x + 3}$: $x - 1 \equiv A(x + 3) + B(x + 5)$

$x = -5$ gives A = 3; $x = -3$ gives B = −2

Integral = [3 ln(x + 5) − 2 ln(x + 3)]

Substitute limits 0 and 2

Answer: (a) 3/(x + 5) + (−2)/(x + 3); (b) 2ln 3 − 5ln 5 + 3ln 7