Day 36 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\cos 5\theta - \cos 3\theta}{\theta^2}$ approaches as $\theta \to 0$.

$\cos 5\theta \approx 1 - \frac{25}{2}\theta^2$ and $\cos 3\theta \approx 1 - \frac{9}{2}\theta^2$

Difference $\approx -8\theta^2$, so the expression ≈ -8

Answer: -8

2 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to show heads with probability 0.25. In 40 trials it does so 13 times.
Test at the 5% level whether the probability is different from 0.25.

H_0: p = 0.25, H_1: p ≠ 0.25 (two-tailed). X ~ B(40, 0.25)

13 is above the mean 10, so use the upper tail: P(X ≥ 13) = 0.179

Each tail is compared with 2.5%: 0.179 is greater than 2.5%

Do not reject H_0: there is insufficient evidence that the probability is different from 0.25

Answer: Do not reject H_0 (p-value 0.179 against 2.5%)

3 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 9 m s^{-1}. 8 s later, car B leaves O from rest with constant acceleration 2.5 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = 9(t + 8); B: distance = ½ × 2.5 × t^2

Equate: $2.5t^2 - 18t - 144 = 0$, so $(t - 12)(2.5t + 12) = 0$

t = 12 (the other root is negative)

OX = 2.5 × 12^2 / 2 = 180 m, and speed = 2.5 × 12 = 30

Answer: (a) 12 s (b) 180 m (c) 30 m s^{-1}

4 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -3$ and $u_{n+1} = \frac{2u_n + 4}{2u_n - 2}$.

Show that the sequence is periodic and find $\sum_{r=1}^{100} u_r$.

$u_1 = -3$, $u_2 = \frac{1}{4}$, $u_3 = -3$.

$u_3 = u_1$, so the sequence repeats with period 2.

One period sums to $-\frac{11}{4}$.

100 = 2 × 50 + 0: 50 full periods plus the first 0 terms.

Answer: $-\frac{275}{2}$

5 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(5, p). It is given that $P(X = 2) = \frac{2}{3} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{20}{2} p^2(1 - p)^3 = \frac{2}{3} \times 5\, p(1 - p)^4$

Divide by $p(1 - p)^3$: $2p = \frac{2}{3}(1 - p)$

$(2 + \frac{2}{3})p = \frac{2}{3}$

$p = \frac{1}{4}$

$P(X = 0) = (1 - p)^5 = (\frac{3}{4})^5 = 0.237$

Answer: (a) $p = \frac{1}{4}$ (b) 0.237

6 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the remainder when x^8 is divided by
$(x - 2)(x + 1)$.

$x^8 = (x - 2)(x + 1)Q(x) + px + q$.
$x = 2$: $256 = 2p + q$.
$x = -1$: $1 = -p + q$.
Subtract: $3p = 255$, so $p = 85$ and $q = 86$.
Answer: 85x + 86

7 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, solve
$\cos 3\theta = 0$ for $0 < \theta < 2\pi$, in terms of π.

Substitute: $\cos\theta(4\cos^2\theta - 3 - 0) = 0$

$\cos\theta = 0$ or $\cos^2\theta = \frac{3}{4}$

$\cos\theta = 0$ gives $\frac{\pi}{2}, \frac{3\pi}{2}$

$\cos^2\theta = \frac{3}{4}$ gives $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Answer: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

8 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = e^x + 2$ to find the exact value of

$\int_0^{\ln 2} \frac{e^x}{e^x + 2}\, dx$

$du = e^x\, dx$; limits $x = 0$, $\ln 2$ become $u = 3, 4$

Integral = $\int_3^4 \frac{1}{u}\, du = [\ln u]$

Evaluate the limits
Answer: $2\ln 2 - \ln 3$

9 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 195 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{5}{12}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{3}{4}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 195\cos\alpha$, so $R = 195 \times \frac{12}{13} - P\sin\beta$

Along the plane: $P\cos\beta = 195\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 195(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{4}{5} + \frac{3}{4} \times \frac{3}{5}) = 195(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13})$, so $P = 168$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_min = 195(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13}) \div \frac{5}{4}$

Answer: (a) 168 N (b) 168 N

10 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|-2x - 6| < -x + 1$.

Boundary values: $-2x - 6 = -x + 1$ gives $x = -7$;

$-2x - 6 = -(-x + 1)$ gives $x = -\frac{5}{3}$.

Also the corner of $|-2x - 6|$ is at $x = -3$.

Testing each region (check the graph of each side) gives the solution set.

Answer: $-7 < x < -\frac{5}{3}$