Day 357 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix}-3\\0\\-3\end{pmatrix} + t\begin{pmatrix}-3\\-1\\-2\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-1\\-3\\3\end{pmatrix} + t\begin{pmatrix}-3\\0\\-2\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}2\\-3\\6\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 0$

$-3 + t(1) = 0$, so $t = 3$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}2\\0\\6\end{pmatrix}$, with length $\sqrt{40}$

Answer: t = 3, minimum distance = $2\sqrt{10}$ m

2 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{3}{5}$, independently.

(a) Find the probability that her fourth score comes on shot number 9.
(b) Find the probability that her fourth score comes on or before shot number 9.

(a) fourth score on shot 9: exactly 3 scores in the first 8 shots, then a score

$C(8, 3)(\frac{3}{5})^3(\frac{2}{5})^5 \times \frac{3}{5}$

$= 0.0743$

(b) On or before shot 9 means at least 4 scores in 9 shots: $X \sim B(9, \frac{3}{5})$

$P(X \geq 4) = 1 - P(X \leq 3) = 0.901$

Answer: (a) 0.0743 (b) 0.901

3 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(8\mathbf{i} + 47\mathbf{j})$ m s^{-1} and $(3\mathbf{i} + 35\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 4 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $8t\mathbf{i} + (47t - 4.9t^2)\mathbf{j}$; position of B: $3t\mathbf{i} + (35t - 4.9t^2)\mathbf{j}$

Subtracting, the $4.9t^2$ terms cancel: $\mathbf{AB} = t(5\mathbf{i} + 12\mathbf{j})$

After 4 s: $\mathbf{AB} = (20\mathbf{i} + 48\mathbf{j})$

Distance $= \sqrt{(20^2 + 48^2)} = 52$

The direction of AB is that of $(5\mathbf{i} + 12\mathbf{j})$, which does not depend on t: tan angle $= \frac{12}{5}$

Answer: (a) 52 m (b) 67.4°

4 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 5 strips to estimate

$\int_1^2 \sqrt{1 + x^3}\, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{5}$; ordinates at $x = 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \ldots, 2$

Sum $= \frac{h}{2}[y_0 + y_5 + 2(y_1 + \ldots + y_4)] = 2.133$

$f''(x) > 0$ on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 2.133; overestimate

5 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $6y = 8x + k$ is a tangent to the circle
$(x + 4)^2 + (y + 1)^2 = 4$.
Find the two possible values of k.

Substitute $y = \frac{8x + k}{6}$: $(x + 4)^2 + (\frac{8x + k}{6} + 1)^2 = 4$.

Multiply by 36 and collect: a quadratic in x with $a = 100$.
Tangent $\Rightarrow$ discriminant = 0. This simplifies to $(k - 26)^2 = 400$.
$k - 26 = \pm 20$.
Answer: k = 6 or k = 46

6 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0 e^{kt}$, t in hours.
Initially P = 500 and after 3 hours P = 1000.
Find k, and the time for the population to reach 1000, to 3 s.f.

$P_0 = 500$. $500\ e^{3k} = 1000$ so $e^{3k} = 2$

$k = \frac{1}{3} \ln 2 = 0.231$

$500\ e^{kt} = 1000$ so $e^{kt} = 2$, $t = \frac{1}{k} \ln 2$

t = 3.00 hours
Answer: k = 0.231, t = 3.00 hours

7 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{12}{x^2 - 4x + 9}$ for all real x.
Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 - 4x + 9 = (x - 2)^2 + 5$.
$(x - 2)^2 \geq 0$, so the denominator is at least 5, with equality at $x = 2$.
The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{12}{5} = \frac{12}{5}$ at $x = 2$.

Answer: greatest value $\frac{12}{5}$ at x = 2; range $0 < f(x) \leq \frac{12}{5}$

8 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 4 and common ratio $\frac{3}{2}$.
Find the least n for which $S_n > 77212$.

$S_n = \frac{4(\frac{3}{2}^n - 1)}{\frac{1}{2}} > 77212$.

$\frac{3}{2}^n > 9652.50$.

$n > \frac{\log(9652)}{\log(1.5)} = 22.628$.

So the least n is 23.
Answer: n = 23

9 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.5 against H_1: p < 0.5 uses a sample of 20 at the 10% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact p = 0.3.

X ~ B(20, 0.5). P(X ≤ 6) = 0.0577 and P(X ≤ 7) = 0.132

The first is at most 0.1 and the second is not, so the critical region is X ≤ 6

Actual significance level = P(X ≤ 6) = 0.0577

Type II error: H_0 is not rejected, so X ≥ 7, with X ~ B(20, 0.3)

P(X ≥ 7) = 1 − P(X ≤ 6) = 0.392

Answer: (a) X ≤ 6 (b) 0.0577 (c) 0.392

10 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 39 m s^{-1}. At the same instant, ball B is released from rest 39 m directly above A.

The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision.

Take g = 9.8 m s^{-2}.

Relative to B, A moves upwards at the constant speed 39 m s^{-1} (both have acceleration g)

So the balls meet when 39 t = 39, giving t = 1

Height of A: $s = 39 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 34.1$

$v = u - gt = 39 - 9.8 \times 1 = 29.2$

Answer: (a) 34.1 m (b) 29.2 m s^{-1} upwards