Day 356 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1000 kg tows a trailer of mass 300 kg along a horizontal road, with driving force 1780 N. Resistances are 500 N on the car and 240 N on the trailer.

(a) Find the tension in the tow bar. At 6.4 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1780 - 500 - 240}{1300} = 0.8$

Trailer: $T - 240 = 300 \times 0.8$, so $T = 480$

After the break the trailer decelerates at $\frac{240}{300} = 0.8$, and stops after $t = \frac{6.4}{0.8} = 8$ s

Trailer travels $\frac{6.4}{2} \times 8 = 25.6$ m

Car: $a = \frac{1780 - 500}{1000} = 1.28$, so $s = 6.4 \times 8 + ½ \times 1.28 \times 8^2 = 92.16$ m

Distance apart = $92.16 - 25.6$

Answer: (a) 480 N (b) 66.56 m

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0 e^{-kt}$ grams, t in days.

Initially m = 60 and after 3 days m = 45.

Find k, and the time for the mass to fall to 3 grams, to 3 s.f.

$m_0 = 60$. $60\, e^{-3k} = 45$ so $e^{-3k} = \frac{3}{4}$

$k = \frac{1}{3} \ln \frac{4}{3} = 0.0959$

$60\, e^{-kt} = 3$ so $e^{kt} = 20$, $t = \frac{1}{k} \ln 20$

t = 31.2 days

Answer: k = 0.0959, t = 31.2 days

3 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 2n + 1$, $u_1 = -5$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 913$.

$u_n = u_1 + \sum_{k=1}^{n-1} (2k + 1) = -5 + 2 \times \frac{(n - 1)n}{2} + 1(n - 1)$.

$= n^2 - 6$.

So a = 1, b = 0, c = −6.

$u_{30} = 894 \leq 913$ and $u_{31} = 955 > 913$.

Answer: (a) a = 1, b = 0, c = −6 (b) n = 31

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4 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £1. Two fair dice are rolled: a total of at least 12 wins £16, a total of at most 3 wins £1, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 12) = $\frac{1}{36}$, P(total ≤ 3) = $\frac{1}{12}$, otherwise $\frac{8}{9}$

Profit: £15 with $\frac{1}{36}$, £0 with $\frac{1}{12}$, £-1 with $\frac{8}{9}$

(a) E = $-\frac{17}{36}$

(b) E(P^2) = $\frac{257}{36}$, Var = $\frac{257}{36} - (-\frac{17}{36})^2 = \frac{8963}{1296}$, so $\sigma = 2.630$

(c) Fair when the stake equals E(winnings) = $\frac{1}{36} \times 16 + \frac{1}{12} \times 1 = \frac{19}{36}$

Answer: (a) $-£\frac{17}{36}$ (b) 2.63 (c) $£\frac{19}{36}$

5 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 585
inclusive that are divisible by 4 or by 5, but not by both.

Multiples of 4: 146 terms (4 to 584): sum = 42924.
Multiples of 5: 117 terms: sum = 34515.
Multiples of both, i.e. of 20: 29 terms: sum = 8700.
Required sum = 42924 + 34515 $-$ 2 × 8700.
Answer: 60039

6 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = y^3 - 2y$.
Find dy/dx and d^2y/dx^2 when y = 1.

dx/dy = $3y^2 - 2 = 1$ when y = 1, so dy/dx = $\frac{1}{dx/dy} = 1$

$d^2x/dy^2 = 6$ when y = 1

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{6}{1^3}$

Answer: dy/dx = 1, $d^2y/dx^2 = -6$

7 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. Find (a) the time of flight, (b) the range, (c) the greatest height. Take g = 9.8 m s^{-2}.

Components: horizontal 14.7, vertical 19.6 (as $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$)

Time of flight: $0 = 19.6\,t - 4.9t^2$, so $t = \frac{19.6}{4.9} = 4$

Range = 14.7 × 4 = 58.8

Greatest height: $0 = 19.6^2 - 2 \times 9.8 \times H$, so $H = \frac{384.16}{19.6} = 19.6$

Answer: (a) 4 s (b) 58.8 m (c) 19.6 m

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (3, 3, 5), (−3, 5, −1), (5, −3, 4).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\text{BA} = \begin{pmatrix} 6 \\ -2 \\ 6 \end{pmatrix}$, $\text{BC} = \begin{pmatrix} 8 \\ -8 \\ 5 \end{pmatrix}$; $\text{BA} \cdot \text{BC} = 94$

$|\text{BA}| = \sqrt{76}$, $|\text{BC}| = \sqrt{153}$

$\cos \text{ABC} = \frac{94}{\sqrt{76}\sqrt{153}} = 0.8717$, so $\text{ABC} = 29.3°$

$|\text{BA} \times \text{BC}| = \left|\begin{pmatrix} 38 \\ 18 \\ -32 \end{pmatrix}\right| = \sqrt{2792}$

$\text{Area} = \frac{1}{2}|\text{BA} \times \text{BC}|$

Answer: ABC = 29.3°, area = $\sqrt{698}$

9 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(18, \frac{1}{3})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{18 - r}{r + 1} \times \frac{1}{2}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(18, r + 1)}{C(18, r)} \times \frac{p}{q} = \frac{18 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{3}$ and $q = \frac{2}{3}$, $p \div q = \frac{1}{2}$

The ratio exceeds 1 when $(18 - r)\frac{1}{3} > (r + 1)\frac{2}{3}$, that is $r < \frac{16}{3}$

So probabilities rise up to r = 6 and then fall; the mode is 6
$P(X = 6) = 0.196$
Answer: 6 with probability 0.196

10 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|-2x + 2| \leq |-4x - 5|$.

Square both sides: $(-2x + 2)^2 \leq (-4x - 5)^2$.
$(2x + 7)(-6x - 3) \leq 0$.

Critical values $x = -\frac{7}{2}$ and $x = -\frac{1}{2}$.

The quadratic has negative coefficient of x^2, so choose the outer region.

Answer: $x \leq -\frac{7}{2}$ or $x \geq -\frac{1}{2}$