Day 352 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(2r + 5) = 247$, find n.

$\sum_{r=1}^{n}(2r + 5) = 2 \times \frac{n(n + 1)}{2} + 5n.$

$\frac{2\,n(n + 1)}{2} + 5n = 247$, so $2n^2 + 12n - 494 = 0$.

Solving $(n > 0)$: $n = 13$.

Answer: n = 13

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(3\theta + 10°) = \frac{\sqrt{2}}{2}$

for $0° \leq \theta \leq 360°$.

Let $u = 3\theta + 10°$; then $\cos u = \frac{\sqrt{2}}{2}$

Base solutions: $u = 45, -45°$ (+ multiples of 360°)

$\theta = (u - 10°) \div 3$, keeping those in range

Answer: θ = 11.7°, 101.7°, 131.7°, 221.7°, 251.7°, 341.7°

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 117.6 m above the ground for 10 s.

Find the speed of projection and the greatest height reached. Take $g = 9.8\ m\ s^{-2}$.

By symmetry, the ball takes 5 s to rise from height 117.6 m to its highest point

Speed at height 117.6 m: $v = 9.8 \times 5 = 49\ m\ s^{-1}$

Speed of projection: $u^2 = v^2 + 2gh = 2401 + 2 \times 9.8 \times 117.6 = 4705.96$

$u = 68.6$. Greatest height $= \frac{u^2}{(2g)} = \frac{4705.96}{19.6} = 240.1$

Answer: 68.6 m s^{-1}, 240.1 m

4 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2 + 1$ is never a multiple of 8, for any integer n.

Consider $n = 8q + r$, with $r = 0, 1, ..., 7$.

Squaring, $n^2 \equiv r^2 \pmod 8$, so:

$0 \to 0; 1 \to 1; 2 \to 4; 3 \to 1; 4 \to 0; 5 \to 1; 6 \to 4; 7 \to 1$ $(r \to n^2 \bmod 8)$

So $n^2 + 1 \equiv 1$ or 2 or 5 (mod 8).

No residue is 0, so $n^2 + 1$ is never a multiple of 8.

Answer: Proved

5 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, $P(A) = 0.6$, $P(B) = 0.2$ and $P(A \mid B) = 0.5$.

(a) Find $P(A \cap B)$.

(b) Find $P(A \mid B')$.

$P(A \cap B) = P(A \mid B) \times P(B) = 0.5 \times 0.2 = 0.1$

$P(A \cup B) = 0.6 + 0.2 - 0.1 = 0.7$

$P(B') = 0.8$

$P(A \cap B') = P(A) - P(A \cap B) = 0.5$

$P(A \mid B') = \frac{0.5}{0.8}$

Answer: (a) 0.1 (b) $\frac{5}{8}$

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6 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = sin x to find the exact value of

$\int_0^{\pi/4} \cos x \sin^4 x \, dx$

du = cos x dx; limits become u = 0 and u = sin(π/4)

Integral = $\int_0^{\sin(\pi/4)} u^4 \, du$

Evaluate the limits

Answer: $\frac{1}{40}\sqrt{2}$

7 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t, y = t^2$. The line $y = 2x - 3$
meets C at A and B. Find the exact length AB.

Substitute: $t^2 = 4t - 3$.
$t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $t = 3$.
A(2, 1), B(6, 9).
$AB^2 = (-4)^2 + (-8)^2 = 80$.
Answer: **AB = $4\sqrt{5}$**

8 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 12 \sin \frac{t}{2} + 5 \cos \frac{t}{2}$.

Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R \sin(\frac{t}{2} + \varphi)$ with $R \cos \varphi = 12$ and $R \sin \varphi = 5$

$R = \sqrt{(12^2 + 5^2)} = 13$ and $\tan \varphi = \frac{5}{12}$, so $\varphi = 0.3948$ rad

Greatest distance = R = 13

$v = \frac{ds}{dt} = 13 \times \frac{1}{2} \cos(\frac{t}{2} + \varphi)$, so greatest speed = $\frac{13}{2}$

s is greatest when $\frac{t}{2} + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 0.3948) \div \frac{1}{2}$

Answer: (a) 13 m (b) $\frac{13}{2}$ m s^{-1} (c) 2.35 s

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 81)$. H_0: $\mu = 31$ is tested against H_1: $\mu < 31$ using the mean of a sample of size 16 at the 1% level.
(a) Find the critical region for the sample mean.
(b) Find the probability of a Type II error when $\mu = 27.5$.

Under H_0, sample mean ~ $N(31, \frac{81}{16})$, standard error = 2.250

z = 2.3263 for the lower tail: critical value = $31 - 2.3263 \times 2.250 = 25.77$
Type II error: do not reject H_0, sample mean > 25.77, where mean ~ N(27.5, 2.250)

$z = \frac{25.77 - 27.5}{2.250} = -0.771$

Probability = 0.780

Answer: (a) sample mean < 25.8 (b) 0.780

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}-5\\-1\end{pmatrix} + t\begin{pmatrix}-3\\-2\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-14\\-8\end{pmatrix} + t\begin{pmatrix}-1\\1\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-9\\-7\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}2\\3\end{pmatrix} = 0$

$-39 + t(13) = 0$, so $t = 3$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-3\\2\end{pmatrix}$, with length $\sqrt{13}$

Answer: t = 3, minimum distance = $\sqrt{13}$ km